Cargese Workshop on Combinatorial Optimization 2022
Open Problems

September 20, 2022

Matthias Walter – Binary linearization complexity

see Matthias' slides below

Christoph Hertrich – Extension complexity and Minkowski sums

Let $P, Q \subseteq \mathbb{R}^d$ be two polytopes and consider their Minkowski sum $P + Q = \{p + q : p \in P,\ q \in Q\}$. It is clear that $\mathrm{xc}(P+Q) \leq \mathrm{xc}(P) + \mathrm{xc}(Q)$ holds. What about the reverse, can we give an upper bound on $\mathrm{xc}(Q)$ in terms of $\mathrm{xc}(P)$ and $\mathrm{xc}(P+Q)$? Do we have $\mathrm{xc}(Q) \leq \mathrm{poly}(\mathrm{xc}(P), \mathrm{xc}(P+Q))$ in general?

Stefan Weltge – Extension complexity and Cartesian products

The question is somewhat similar to Christoph's question. Let $P, Q \subseteq \mathbb{R}^d$ be two polytopes and consider their Cartesian product $P \times Q$. It is clear that $\mathrm{xc}(P \times Q) \leq \mathrm{xc}(P) + \mathrm{xc}(Q)$ holds. Does $\mathrm{xc}(P \times Q) = \mathrm{xc}(P) + \mathrm{xc}(Q)$ hold for all P and Q?

Manuel Aprile – Number of variables of an extended formulation

For a polyhedron $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ and for an integer $k \geq \mathrm{xc}(P)$, let $\mathrm{xd}(P, k)$ denote the minimum number of variables in an extended formulation of P of size at most k (i.e. with at most k inequalities).

- What can we say about $\mathrm{xd}(P, k)$ in general? Conjecture: $\mathrm{xd}(P, k)$ is equal to the minimum of $k - \mathrm{rank}(T)$, where T has at most k columns and is the first factor in a non-negative factorization of the slack matrix of P. (This is motivated by the proof of Yannakakis' Theorem).
- Let P be the spanning tree polytope of the complete graph on n vertices. Is there an extended formulation of P with $O(n^3)$ inequalities and $O(n^2)$ variables (or anything that is $o(n^3)$)?

Volker Kaibel – Correlation polytope

Let $\mathrm{COR}(n) := \mathrm{conv}\{xx^\intercal : x \in \{0,1\}^n\}$ denote the correlation polytope, which is isomorphic to the cut polytope of the complete graph on $n+1$ nodes. Since the number of vertices of $\mathrm{COR}(n)$ is 2^n, we obtain the upper bound $\mathrm{xc}(\mathrm{COR}(n)) \leq 2^n$ for all n. The seminal work of Fiorini, Massar, Pokutta, Tiwary & de Wolf (2012) shows that there is some $\varepsilon > 0$ such that $\mathrm{xc}(\mathrm{COR}(n)) \geq (1+\varepsilon)^n$ for all n. Later, Kaibel & Weltge (2013) showed that $\mathrm{xc}(\mathrm{COR}(n)) \geq 1.5^n$ holds for all n.

- Can we improve the lower bound even further?
- Actually, does $\mathrm{xc}(\mathrm{COR}(n)) \leq 2^n - 1$ hold for some n?

Mathieu Van Vyve – Extension complexity of a chance-constrained model

see Mathieu's slides below

Samuel Fiorini – Converting a totally Δ-modular matrix into a TU matrix

Let $A \in \mathbb{Z}^{m \times n}$ be an integer matrix without repeated columns or rows such that all determinants of square submatrices (of any size) of A are in $\{-\Delta, \dots, \Delta\}$. Can we remove $o(m+n)$ rows and/or columns from A to obtain a totally unimodular matrix?

Laura Vargas Koch – Solidarity cover

Given a metric space (X, d), an (m, r)-solidarity cover is a partition of the points X into pairwise disjoint sets $X_1, ..., X_m$ such that $\{p \in X : d(p, X_i) \leq r\} = X$ for all $i \in [m]$. It is NP-hard to decide whether an (m, r)-solidarity cover exists. Now assume there exists an (m^*, r^*)-solidarity cover for some numbers m^*, r^*. We can aim for approximation algorithms in two different ways. Either, we can fix r^* and aim to compute a $(\beta m^*, r^*)$-solidarity cover for some $\beta < 1$ or we fix m^* and do the analogue, i.e., compute an $(m^*, \alpha r^*)$-solidarity cover for some $\alpha > 1$. We have partial results for these problems in general metric spaces as well as in 2D but there are still huge gaps.

Andreas Emil Feldmann – Extension complexity of representative vectors

Let k be a positive integer and let $e_S \in \mathbb{R}^{2^k}$ represent the set $S \subseteq [k]$, i.e. the entry is 1 at the position for set S and 0 everywhere else. Let $P = \text{conv}\{(x, y, z) : x = e_S, y = e_{S'} \text{ with } S \cap S' \neq \emptyset \text{ and } z = e_{S \cup S'}\} \subseteq \mathbb{R}^{3 \cdot 2^k}$. It is easy to see that $\text{xc}(P) \leq 3^k$ holds. Does $\text{xc}(P) \leq 2^k$ hold? The motivation to answer this question comes from algorithmic questions regarding Steiner trees.

Daniele Catanzaro – Extension complexity of the BMEP polytope

Let $\Gamma = \{1, \dots, n\}$ denote a set of $n \geq 4$ vertices and let D denote a given $n \times n$ distance matrix whose generic entry d_{ij} encodes a distance between the pair of vertices $i, j \in \Gamma$. Then, the Balanced Minimum Evolution Problem (BMEP) consists of finding an Unrooted Binary Tree (UBT) T^* having Γ as a leafset and such that the length function

$$L(T) = \sum_{i \in \Gamma} \sum_{j \in \Gamma} \frac{d_{ij}}{2^{\tau^*_{ij}}}$$

is minimized, where τ^*_{ij} is the path-length in T^* between two leaves $i, j \in \Gamma$.

- What is the extension complexity of the BMEP polytope?
- How many operations are necessary to transform a caterpillar UBT of Γ into a Balanced UBT of Γ?

Sophie Huiberts – TSP with unknown distances

Suppose we want to solve an instance of TSP for which the distances on the arcs are unknown. We can query the distance between any two vertices within a factor of $(1 + \epsilon)$ by paying an extra cost of $\frac{1}{\epsilon}$.

- Which arcs should we purchase?
- What constraints are necessary to find approximation algorithms? Maybe assumptions like metric distances?

Similar questions can be asked for other combinatorial optimization problems for which the objective function is initially unknown.

Appendix: Matthias' slides

Application

Idea · Low-hanging fruits · Questions

Problem – Polynomial binary optimization

Minimize a given polynomial $f(x)$ over $x \in \{0,1\}^n$.

Generic form:

$$f(x) = \sum_{I \subseteq [n]} c_I \prod_{i \in I} x_i$$

Approach via multilinear polytopes:

1. Replace every monomial $\prod_{i \in I} x_i$ with a nonzero c_I by an auxiliary variable z_I.
2. Add linearization constraints that ensure $z_I = \prod_{i \in I} x_I$ for each such monomial:

$$z_I \leq x_i \qquad \forall i \in I \tag{1a}$$

$$1 - z_I \leq \sum_{i \in I} (1 - x_i) \tag{1b}$$

3. Note that $f(x)$ is linear in z: $\sum_{I \subseteq [n]} c_I z_I$

Observation – Practical instances

Some of the (few) existing benchmark instances have $c_I \neq 0$ for quite many $I \subseteq [n]$.

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Can We Do Better?

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Definition – Binary linearization

Let $f : \{0,1\}^n \to \mathbb{R}$. A **binary linearization of** f is given by $a \in \mathbb{R}^k$, $g_1, g_2, \ldots, g_k : \{0,1\}^n \to \{0,1\}$ and $\beta \in \mathbb{R}$ such that

$$f(x) = \sum_{i=1}^{k} a_i g_i(x) + \beta \quad \forall x \in \{0,1\}^n \tag{2}$$

holds. Its **size** is the number k of summands.

Why binary functions?

- Some restriction is needed: otherwise, $g_1 := f$ allows $k = 1$.
- Binaries are nice: one can always model $y = g(x)$ for all $x \in \{0,1\}^n$ using linear inequalities.

Definition – Binary linearization complexity

Let $f : \{0,1\}^n \to \mathbb{R}$ and let $\mathcal{G}$ be a family of binary-valued functions $g : \{0,1\}^n \to \{0,1\}$. The **binary linearization complexity of** f **over** $\mathcal{G}$, denoted by $blc_{\mathcal{G}}(f)$ is the minimum size of a binary linearization that uses only $g_i \in \mathcal{G}$.

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Example & Low-hanging Fruits

Idea · Low-hanging fruits · Questions

Some classes of functions:

$$\mathcal{P} := \{\text{all binary-valued functions}\} \qquad \mathcal{M} := \{g(x) = \prod_{i \in I} x_i\} \qquad \mathcal{C} := \{g(x) = \prod_{i \in I} x_i \cdot \prod_{j \in J} (1 - x_j)\}$$

Example:

$f(x) = -x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 = (1 - x_1) \cdot (1 - x_2) \cdot (1 - x_3) - 1$ implies $blc_{\mathcal{C}}(f) = 1$.

Proposition – Trivial upper bound

Let $\mathcal{G} \supseteq \mathcal{M}$ and $f : \{0,1\}^n \to \mathbb{R}$ be arbitrary. Then $blc_{\mathcal{G}}(f) \leq 2^n$.

Lemma – Monotonicity

Let $\mathcal{G}_1 \subseteq \mathcal{G}_2$ and $f : \{0,1\}^n \to \mathbb{R}$ arbitrary. Then $blc_{\mathcal{G}_1}(f) \geq blc_{\mathcal{G}_2}(f)$.

Proposition – Monomials

For $f : \{0,1\}^n \to \mathbb{R}$, $blc_{\mathcal{M}}(f)$ is the minimum number of monomials of a polynomial $f' \equiv f$ (on $\{0,1\}^n$).

Theorem – Random functions

Let $\mathcal{G} \subseteq \mathcal{P}$ and $f : \{0,1\}^n \to \mathbb{R}$ be random, then $blc_{\mathcal{G}}(f) \geq 2^n$ almost surely.

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Open questions:

1. Are there any lower bounds on $blc_{\mathcal{C}}(f)$?
2. Are there any upper bounds on $blc_{\mathcal{C}}(f)$?
3. What is the complexity of computing $blc_{\mathcal{G}}(f)$ for certain $\mathcal{G}$s (after defining how f is given)?
4. Can one approximate $blc_{\mathcal{C}}(f)$, i.e., construct reasonably-sized binary linearizations?
5. With which function should we enhance $\mathcal{C}$ to get small $blc(f)$? (e.g., parity indicator, indicator whether cardinality has a specific value, ...)

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Appendix: Mathieu's slides

A simple chance-constrained model:

Luedtke, Ahmed, Nemhauser, *Math.Prog.A* (2010)
An integer programming approach for linear programs with probabilistic constraints

$$G' = \left\{ (y, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{n} z_i \leq p, \; y + (h_i - h_{p+1}) z_i \geq h_i \quad i = 1, \ldots, n \right\}.$$

A natural extended formulation:

$$y + \sum_{i=1}^{p} (h_i - h_{i+1}) w_i \geq h_1$$
$$w_i - w_{i+1} \geq 0 \quad i = 1, \ldots, p$$
$$z_i - w_i \geq 0 \quad i = 1, \ldots, p$$
$$\sum_{i=1}^{n} z_i \leq p.$$

But this is not tight, you need an additional exp family for that:

$$\sum_{i \in S} z_i + \sum_{i \in \{k, \ldots, p\} \setminus S} w_i \leq p - k + 1 \qquad \forall S \in \mathcal{S}_k, \; k = 1, \ldots, p.$$

$$\mathcal{S}_k = \{S \subseteq \{k, \ldots, n\} : |S| \leq p - k + 1\}$$

What is the extension complexity of conv(G')?

And what is its description in the original variable space