Packing and Covering: Lecture 1

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The Set Covering problem

Let A be a 0,1 matrix.

The *set covering problem* is the following optimization problem:

$$\begin{aligned} &\text{Min } cx \\ &Ax \geq \mathbf{1} \\ &x \in \{0,1\}^n \end{aligned}$$

EXAMPLE OF APPLICATION:

Airline crew scheduling: the rows of A are flight legs to cover, and the columns represent possible "rotations" for the crews.

QUESTIONS:

1) When is the polyhedron $Ax \geq \mathbf{1},\ x \geq 0$ integral?

2) When is the linear system $Ax \geq \mathbf{1},\ x \geq 0$ totally dual integral?

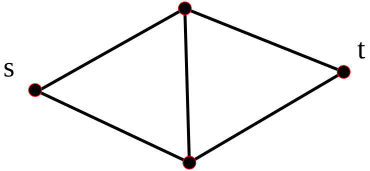
Ideal Matrices

A 0,1 matrix A is *ideal* if all the extreme points of the polyhedron

$$\begin{aligned} Ax &\geq \mathbf{1} \\ x &\geq 0 \end{aligned}$$

are 0,1 vectors.

EXAMPLE: Consider a graph with source s and sink t. Let A be a 0,1 matrix whose columns are indexed by the edges of the graph and whose rows are the characteristic vectors of the st-paths.

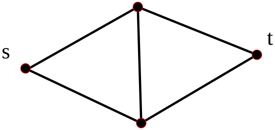


$$A := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Fulkerson 1970 proved that the extreme points of $Ax \geq \mathbf{1}$, $x \geq 0$ are the characteristic vectors of the st-cuts.

Blocker

Let A be a 0,1 matrix with no row dominating another. The 0,1 vectors x satisfying $Ax \geq \mathbf{1}$ are called *covers* of A. The *blocker* of A is the 0,1 matrix $b(A)$ whose rows are the minimal covers of A.



$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

st-paths

$$b(A) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

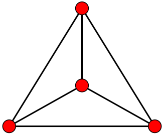
st-cuts

REMARK $b(b(A)) = A$

THEOREM Lehman 1965: A is ideal if and only if $b(A)$ is ideal.

EXAMPLES: st-paths, st-cuts, T-joins, T-cuts, dijoins, dicuts.

ANOTHER IDEAL EXAMPLE: The triangles of K_4



$$A := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Proof of Lehman's Theorem

THEOREM Lehman 1965:
A 0,1 matrix A is ideal if and only if $b(A)$ is ideal.

PROOF Because $b(b(A)) = A$, it suffices to prove that A ideal implies $b(A)$ ideal.
Let a be an extreme point of $Q := \{y : \; b(A)y \geq \mathbf{1}, \; y \geq 0\}$.
Because A is ideal, the rows of $b(A)$ are exactly the extreme points of $P := \{x : \; Ax \geq \mathbf{1}, \; x \geq 0\}$.
Therefore $ax^T \geq 1$ holds for all the extreme points $\bar{x}$ of P.
Since $a \geq 0$, $ax^T \geq 1$ holds for every point x in P.
Furthermore $ax^T = 1$ for some $x \in P$. By LP duality,

$$1 = \min\{a^T x : \; x \in P\} \quad = \quad \max\{\sum \lambda_i : \; \lambda^T A \leq a^T, \; \lambda \geq 0\}.$$

That is, a belongs to the up-monotone hull of the rows of A. Call this integral polyhedron Q_I.
Since $a \in Q_I$ is an extreme point of Q and $Q_I \subseteq Q$, a is a 0,1 extreme point of Q.

Primal AND dual integrality

Let A be an $m \times n$ 0,1 matrix. The matrix A *packs* if

$$\begin{array}{ccc} \tau := \text{Min } \sum_j x_j & = & \nu := \text{Max } \sum_i y_i \\ Ax \geq \mathbf{1} & & yA \leq \mathbf{1} \\ x \in \{0,1\}^n & & y \in \{0,1\}^m \end{array}$$

THEOREM Menger 1927: The minimum size of an *st*-cut equals the maximum number of edge-disjoint *st*-paths.

THEOREM Lucchesi-Younger 1978: The minimum size of a dijoin equals the maximum number of edge-disjoint dicuts.

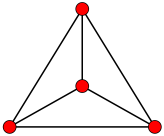
CONJECTURE Woodall 1978: The minimum size of a dicut equals the maximum number of edge-disjoint dijoins. **More on this in Lecture 2**

Ideal $0,1$ Matrix that does not Pack

Recall that the $0,1$ matrix A *packs* if

$$\tau := \text{Min } \sum_j x_j \quad = \quad \nu := \text{Max } \sum_i y_i$$
$$Ax \geq \mathbf{1} \qquad\qquad yA \leq \mathbf{1}$$
$$x \in \{0,1\}^n \qquad\qquad y \in \{0,1\}^m$$

Example of a 0,1 matrix that does not pack: The triangles of K_4



$$A := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Any two rows of A intersect ($\nu = 1$), but no single column covers all the rows ($\tau \geq 2$).

CONJECTURE Seymour 1975: An ideal 0,1 matrix has a $1/2^k$-integral packing y of value τ. Seymour asked whether $1/4$-integral packings might suffice. **More on this in Lecture 3**

Minors

In the set covering formulation $Ax \geq \mathbf{1}$, $x \geq 0$, setting $x_j = 0$ corresponds to removing column j of matrix A; setting $x_j = 1$ corresponds to removing column j and all the rows with a 1 in column j.

EXAMPLE: For the matrix of st-paths, these operations correspond to contracting and deleting edges.

Any matrix obtained from A by a sequence of these two operations is called a *minor* of A.

REMARK If a $0,1$ matrix is ideal, then also all its minors are ideal.

NOTE: It is standard to remove from A and its minors any row a' such that $a' \geq a$ for some other row a since the inequality $a'x \geq 1$ is implied by $ax \geq 1$.

The packing property

The property that a 0,1 matrix packs is not preserved under taking minors.

For example, $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

A 0,1 matrix A has the *packing property* if it packs and all its minors pack.

EXAMPLE: For the matrix of st-paths, taking minors corresponds to contracting and deleting edges. It follows from Menger's theorem that the matrix of st-paths has the packing property.

THEOREM Lehman 1990
If a 0,1 matrix has the packing property, it is ideal.

The converse is not true, as shown by the triangles of K_4.

Total dual integrality

CONJECTURE Conforti and Cornuéjols (1993)

A 0,1 matrix A has the packing property if and only if the system $Ax \geq \mathbf{1}$, $x \geq 0$ is totally dual integral, namely

$$\begin{array}{ccc} \text{Min } cx & = & \text{Max } \sum_i y_i \\ Ax \geq \mathbf{1} & & yA \leq \text{c} \\ x \in \{0,1\}^n & & y \in \mathbb{Z}_+^m \end{array}$$

for all $c \in \mathbb{Z}_+^n$.

REMARK It follows from the definition that A has the packing property if and only if equality holds above for all $c \in \{0, 1, n\}^n$.

The above conjecture is equivalent to the *Replication Conjecture*: If equality holds for all $c \in \{0, 1, n\}^n$, it holds for $c = (2, 1, \ldots, 1)$.

Analogy with perfect matrices

An $m \times n$ 0,1 matrix A is *perfect* if, for every column submatrix $\bar{A}$

$$\begin{array}{ccc} \text{Max } \sum_j x_j & = & \text{Min } \sum_i y_i \\ \bar{A}x \leq \mathbf{1} & & y\bar{A} \geq \mathbf{1} \\ x \in \{0,1\}^n & & y \in \{0,1\}^m \end{array}$$

NOTE If A is the StableSet-Node incidence matrix of a graph G, the above equality says: Max clique = chromatic number, for G and all its induced subgraphs, i.e. G is a perfect graph.

THEOREM Lovász 1972

The following properties are equivalent for a 0,1 matrix A.

(i) The matrix A is perfect

(ii) the linear system $Ax \leq \mathbf{1}$, $x \geq 0$ is totally dual integral

(iii) the polytope $Ax \leq \mathbf{1}$, $x \geq 0$ only has 0,1 extreme points.

Note the analogy between (i) $\Leftrightarrow$ (ii) and the previous conjecture.

Lovász proved this theorem by first proving a "Replication Lemma".

Minimally imperfect matrices

A $0,1$ matrix A is *minimally imperfect* if it is not perfect but all its proper column submatrices are perfect.

THEOREM Chvátal, Fulkerson, Lovász 1972
Chudnovsky, Robertson, Seymour, Thomas 2006
A $0,1$ matrix A is minimally imperfect if and only if it is the matrix

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & & 1 \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & 1 & & 0 \end{pmatrix}$$

or it is the clique-node matrix of an odd hole or of the complement of an odd hole.

Differences between perfection and idealness

THEOREM Chudnovsky, Cornuéjols, Liu, Seymour, Vuskovič 2005

Given a 0,1 matrix A, there is a polynomial algorithm to decide whether A is perfect.

THEOREM Ding, Feng, Zang 2008

Given a 0,1 matrix A, it is co-NP-complete to decide whether A is ideal.

The difference boils down to: there is a polynomial algorithm to decide whether a graph contains an odd hole Chudnovsky, Scott, Seymour, Spirkl 2020, whereas it is co-NP-complete to decide whether a 0,1 matrix contains an odd hole minor (Ding, Feng, Zang 2008).

ANOTHER DIFFERENCE

For a 0,1 perfect matrix A, the system $Ax \leq \mathbf{1}$, $x \geq 0$ is TDI.
For a 0,1 ideal matrix A, the system $Ax \geq \mathbf{1}$, $x \geq 0$ is not always TDI.

Minimally nonideal 0,1 matrices

A 0,1 matrix A is *minimally nonideal* if it is not ideal but all its minors are ideal.

THEOREM Lehman 1990

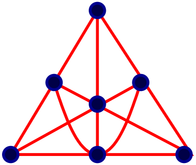
An $m \times n$ 0,1 matrix A is minimally nonideal if and only if it is

$$\Delta_n := \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 1 & 0 & 0 & & 1 \end{pmatrix}$$

or its rows of minimum cardinality form a square submatrix $\bar{A}$ satisfying $\bar{A}B = E + kI$ for some square 0,1 matrix B and $k \geq 1$.

Examples of minimally nonideal matrices

- Δ_n
- Odd holes and their blocker
- The point-line incidence matrix of the Fano plane:



$$A := \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- The odd cycles of K_5 (versus the edges of K_5, i.e. 10 columns)
- D_8 due to Guoli Ding
- A new infinite family found by Jonathan Wang 2011

Idealness and k-wise intersecting 0,1 matrices

A 0,1 matrix A is k-wise intersecting if any k rows have a column of all 1's, but the matrix A itself does not have a column of all 1's.

CONJECTURE A 4-wise intersecting 0,1 matrix is not ideal.

THEOREM Abdi, Cornuéjols, Huynh, Lee 2022
A 4-wise intersecting 0,1 matrix with the binary property is not ideal.

Binary property means that for any 3 rows a_1, a_2, a_3, there exists a row a such that $a_1 + a_2 + a_3 \pmod 2 \geq a$. Examples: st-paths, st-cuts, T-joins, T-cuts.

PROPOSITION Abdi, Cornuéjols, Huynh, Lee 2022
There exists an ideal 3-wise intersecting 0,1 matrix with the binary property.

A 64x30 example is constructed from the Petersen graph as follows: subdivide every edge; let T be any nodeset of even cardinality that contains all the new nodes; take the incidence matrix of the T-joins.

Minimally nonpacking 0,1 matrices

A 0,1 matrix A is *minimally nonpacking* if it does not pack but all its minors pack.

Example: The triangles of K_4.

PROPERTY (a consequence of Lehman's theorem)
A minimally nonpacking matrix is either minimally nonideal or ideal.

Minimally nonideal minimally nonpacking 0,1 matrices

Let A be a minimally nonideal 0,1 matrix different from Δ_n. Recall that, by Lehman's theorem, the rows of A of minimum cardinality form a square submatrix $\bar{A}$ satisfying $\bar{A}B = E + kI$ for some square $0,1$ matrix B and $k \geq 1$.

THEOREM Cornuéjols, Guenin, Margot 2000
If A is minimally nonpacking, then $k = 1$.

CONJECTURE Cornuéjols, Guenin, Margot 2000
A is minimally nonpacking if and only if $k = 1$.

Examples of ideal minimally nonpacking matrices

- The triangles of K_4 Lovász 1972; Seymour 1977 conjectured that this is the only example. But...
- Another example was found by Schrijver in 1980.
- Cornuéjols, Guenin and Margot 2000 found an infinite class and a few other small examples.
- Abdi, Cornuéjols, Guričanová and Lee 2018 found over 700 other small examples ($n \leq 14$).
- Abdi, Cornuéjols, Lee and Superdock 2019 found another infinite class.

CONJECTURE (the $\tau = 2$ Conjecture)
Cornuéjols, Guenin, Margot 2000
Every ideal mnp matrix has covering number $\tau = 2$.

THEOREM This conjecture implies the Replication Conjecture.

Co-exclusive columns (Abdi, Fukasawa and Sanità MOR 2017)

We say distinct columns j, k are *co-exclusive* in a $0{,}1$ matrix A if every minimal $0{,}1$ solution x of $Ax \geq 1$ satisfies $x_j + x_k \leq 1$.

The *identification* $A|_{j=k}$ is the $0{,}1$ matrix with column k removed whose rows are obtained from those of A by setting $a_{ij}|_{j=k} = 0$ if $a_{ij} = a_{ik} = 0$ in A, and setting $a_{ij}|_{j=k} = 1$ otherwise.

PROPERTY If A does not pack, then neither does $A|_{j=k}$.

Thus, identification reduces one non-packing matrix to a smaller one, so let us add this operation to the operation of taking minors.

Ideal mnp matrices with co-exclusive columns

THEOREM Abdi, Cornuéjols, Pashkovich MOR 2018

Let A be an ideal minimally non-packing 0,1 matrix with co-exclusive edges j, k. Then either
(i) $A|_{j=k}$ is another ideal minimally non-packing 0,1 matrix, or
(ii) $A|_{j=k}$ is not minimally non-packing, and every minimally non-packing minor has covering number 2.

This stresses the importance of ideal minimally non-packing 0,1 matrices with covering number 2.

Intersecting minors

We say that a 0,1 matrix is *intersecting* if the supports of any two rows have nonempty intersection, but A has no column of all 1's.

CONJECTURE Abdi, Cornuéjols and Lee 2018
Let A be a 0,1 matrix. If A has no intersecting minor, then A is ideal if and only if the linear system $Ax \geq \mathbf{1}$, $x \geq 0$ is totally dual integral.

Equivalently: For $c \in \mathbb{Z}^n_+$ consider the dual pair of linear programs

$$(P)\ \text{Min } cx \quad = \quad (D)\ \text{Max } \sum_i y_i$$
$$Ax \geq \mathbf{1} \qquad\qquad yA \leq c$$
$$x \in \{0,1\}^n \qquad\qquad y \in \mathbb{Z}^m_+$$

CONJECTURE If A has no intersecting minor, the following statements are equivalent:

(i) (P) has an integral optimal solution for all $c \in \mathbb{Z}^n_+$,

(ii) (D) has an integral optimal solution for all $c \in \mathbb{Z}^n_+$.

Checking whether a 0,1 matrix has an intersecting minor

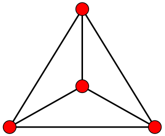
PROPOSITION Abdi, Cornuéjols and Lee 2018
The above conjecture is equivalent to the $\tau = 2$ conjecture.

THEOREM Abdi, Cornuéjols and Lee 2018
One can check in polynomial time whether a 0,1 matrix A has an intersecting minor.

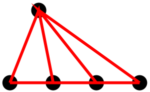
THEOREM Drees IPCO 2022 One can check in polynomial time whether a 0,1 matrix A has a k-wise intersecting *restriction*.

A *restriction* is a minor of A where some columns are deleted and all resulting columns of all 1's are contracted.

Examples of intersecting 0,1 matrices



$$Q_6 := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$



$$\Delta_n := \begin{pmatrix} 1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ \dots & & & & & \dots \\ 0 & 1 & 1 & 1 & \dots & 1 \end{pmatrix}$$

BlockerExtendedOddHole

$$:= \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ \dots & & & & & & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Neither a delta nor the blocker of an extended odd hole has a fractional packing of value 2.

Finding a delta or the blocker of an extended odd hole minor

A converse of this statement holds:
THEOREM Abdi and Lee 2018
A 0,1 matrix with $\tau \geq 2$ and no fractional packing of value 2 has a delta or the blocker of an extended odd hole minor.

THEOREM Abdi, Cornuéjols and Lee 2018
Given a 0,1 matrix, one can find a delta or the **blocker** of an extended odd hole minor, or certify that none exists, in polynomial time (Drees IPCO 2022 improved the complexity).

This is surprising in view of:
THEOREM Ding, Feng, Zang 2008
Given a 0,1 matrix A, it is co-NP-complete to decide whether A has a delta or an extended odd hole minor.