Lecture 2:
Some Thoughts on Woodall's Conjecture

Ahmad Abdi LSE
Gérard Cornuéjols CMU
Michael Zlatin CMU

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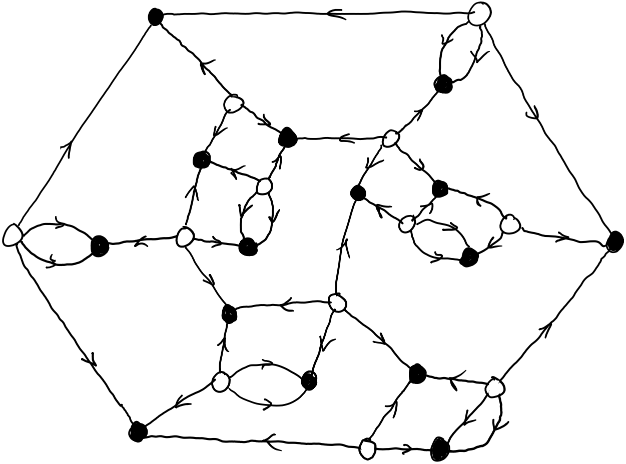
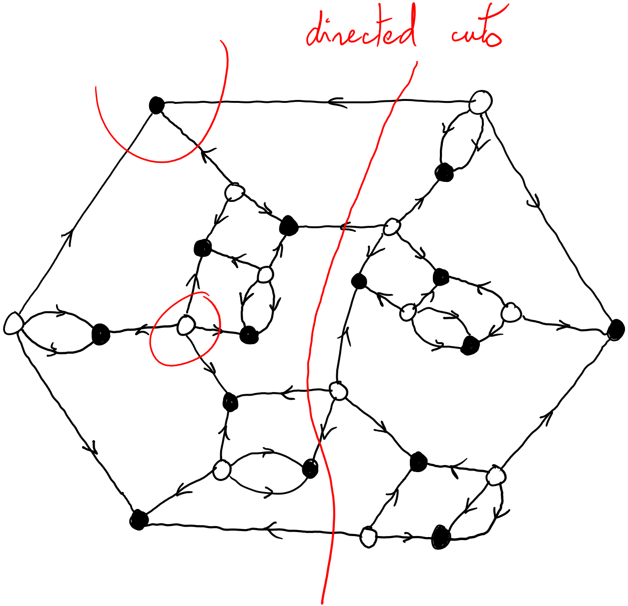
Dijoins and directed cuts

A digraph

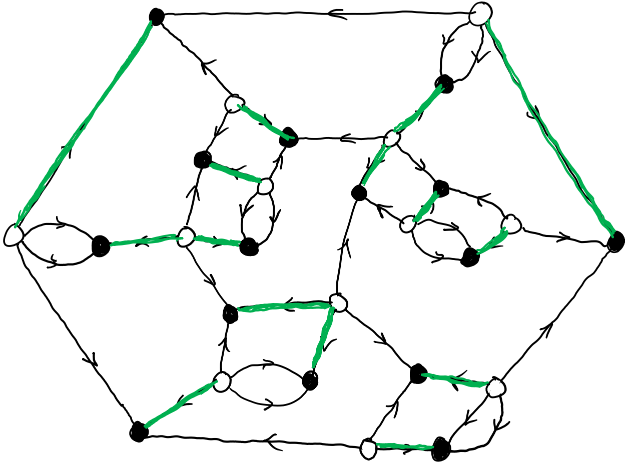


Dijoins and directed cuts

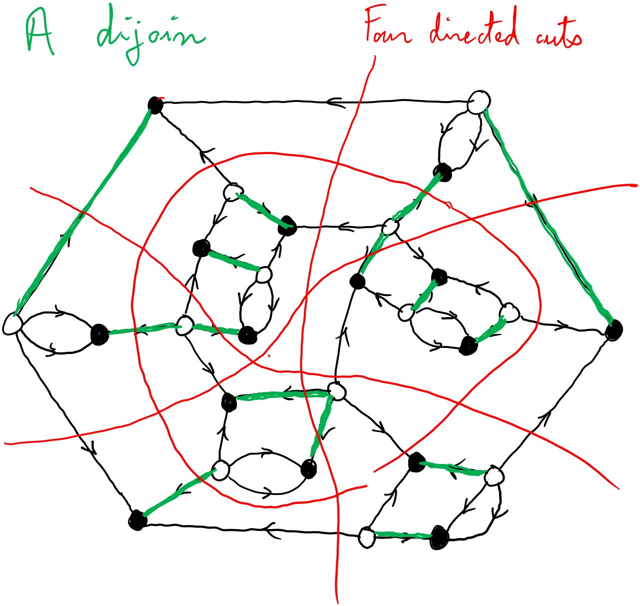


Dijoins and directed cuts

A dijoin



Dijoins and directed cuts



Observations

REMARK In a connected digraph D, the dijoins are precisely the arc sets that, when contracted, make the resulting digraph strongly connected.

PROOF Consider a digraph D' obtained by contracting arcs of D. Any dicut in D' is also a dicut of D.
It follows that, contracting a dijoin destroys all the dicuts.
Conversely, consider any arc set J that, when contracted, makes the resulting digraph D' strongly connected. Then J must contain at least one arc from every dicut of D. So J is a dijoin.

LEMMA A connected bridgeless digraph D contains 2 disjoint dijoins.

PROOF Let G the underlying graph of D. G is 2-edge connected. Therefore G admits a strongly connected orientation O.
Let J_1 be the arcs of D whose orientations coincide with O, and J_2 those arcs where the orientation differs. Both J_1 and J_2 intersect any dicut $F = (U, \bar{U})$ of D. Indeed, if J_1 contains all the arcs in F, so does O, contradicting that O is strongly connected. If J_2 contains all the arcs in F, then all arcs of O are directed from $\bar{U}$ to U, again a contradiction.

The Lucchesi-Younger Theorem (1978)

Consider a digraph $D = (V, E)$ with arc weights $w \in \mathbb{Z}_+^E$.

The minimum weight of a dijoin is equal to the maximum number of dicuts such that each arc a belongs to at most w_a of them.

Let A be the dicut-arc incidence matrix. This theorem states that the linear system $Ax \geq \mathbf{1},\ x \geq 0$ is totally dual integral.

By Edmonds-Giles 1977, this implies that matrix A is ideal.

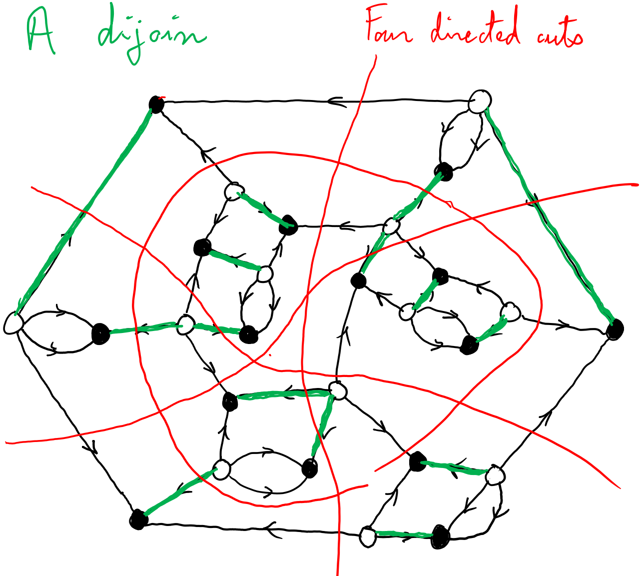
This, together with Lehman's theorem that we proved in Lecture 1, implies that the dijoin-arc incidence matrix is also ideal. Therefore :

The minimum cardinality τ of a dicut is equal to the maximum value of a fractional packing of dijoins.

Woodall's conjecture states that there is an integral packing of dijoins of value τ.

Woodall's conjecture

Are there 3 disjoint dijoins in this digraph ?



Woodall's conjecture

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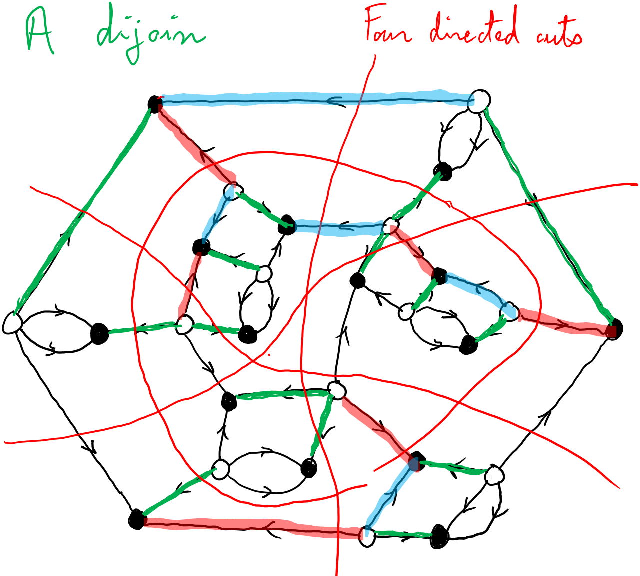

Woodall's conjecture

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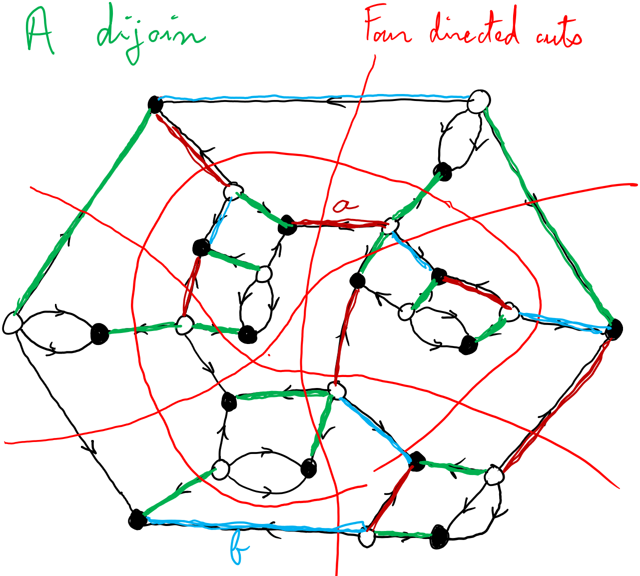


Woodall's conjecture

Are there 3 disjoint dijoins in our example digraph ?

Actually, there are.

But the green dijoin cannot be used in such a packing.

Why can't the green dijoin be used in a packing ? Not clear...

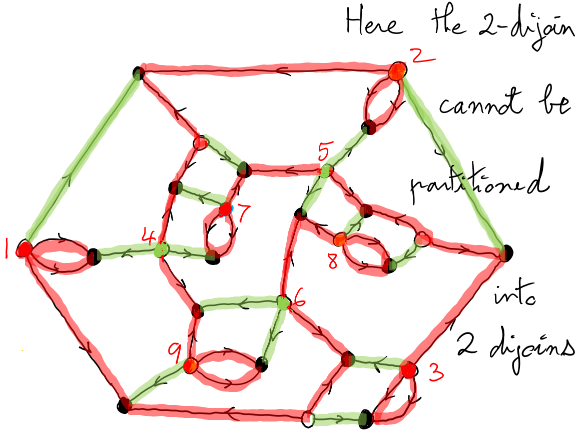
This points to the difficulty of proving Woodall's conjecture.

We can prove a weaker statement.

A dijoin and a $\tau - 1$ dijoin

In a digraph D, a k-dijoin is a set of arcs that intersects every dicut at least k times.

THEOREM Abdi, Cornuejols, Zlatin The arcs of D can be partitioned into a dijoin and a $\tau - 1$ dijoin, where τ is the smallest size of a dicut.



Prior work on Woodall's Conjecture

Frank and Tardos 1984 formulate the conjecture as a common basis packing problem for two matroids.

Schrijver 1980 ; Feofiloff and Younger 1987 proved the conjecture for source-sink connected digraphs.

Lee and Wakabayashi 2001 proved the conjecture for series-parallel digraphs.

Lee and Williams 2006 proved the conjecture for digraphs without a $K_5 \setminus e$ minor.

Mészáros 2018 proved the conjecture for digraphs that are $(q-1, 1)$-partition conected for q a prime power.

Shepherd and Vetta 2005 consider half-integral packings of dijoins.

Chudnovsky, Edwards, Kim, Scott, Seymour 2016 study a connectedness condition.

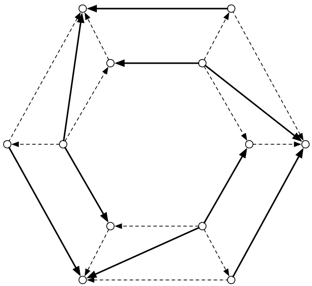
The Weighted Case

Consider a digraph $D = (V, E)$ with arc weights $w \in \mathbb{Z}_+^E$.
Let A be dijoin-arc incidence matrix. By weak LP duality

$$\begin{array}{ccc} \tau_w := \text{Min } \sum_j w_j x_j & \geq & \nu_w := \text{Max } \sum_i y_i \\ Ax \geq \mathbf{1} & & yA \leq w \\ x \in \{0,1\}^n & & y \text{ integral.} \end{array}$$

CONJECTURE Edmonds and Giles 1977 $\tau_w = \nu_w$.

Counterexample
Schrijver 1980



Harvey, Király, Lau 2011 This is false for any $\tau_w \geq 2$.

Reduction to "almost" regular bipartite digraphs (Abdi, Cornuejols, Zlatin)

Let D be a digraph where every dicut has size at least τ.

Woodall's conjecture states that D contains τ disjoint dijoins.

Recall that Woodall's conjecture is true for $\tau = 2$.

THEOREM
To prove Woodall's conjecture for $\tau \geq 3$, it is sufficient to prove it for digraphs where all nodes are sources or sinks, all sinks have degree τ, and all sources have degree τ or $\tau + 1$.

Properties of our reduction

Let D be a digraph where every dicut has size at least $\tau \geq 3$.
We may assume that D has no cut vertex.

We construct a bipartite digraph B where all nodes are sources or sinks, all sinks have degree τ, and all sources have degree τ or $\tau + 1$.

1. The digraph D is obtained from B by contracting some of its arcs.
2. If D is planar, B is also planar.
3. Packing dijoins in D becomes closer to the problem of packing perfect matchings in B. We will elaborate on this later.
4. Our reduction also works for weighted digraphs D.

The ρ parameter

For $v \in V$, let $m_v \in \{0, 1, \ldots, \tau - 1\}$ such that $m_v \equiv |\delta^+(v)| - |\delta^-(v)| \pmod{\tau}$. Let

$$\rho := \frac{1}{\tau} \sum_{v \in V} m_v.$$

Another property of our reduction :

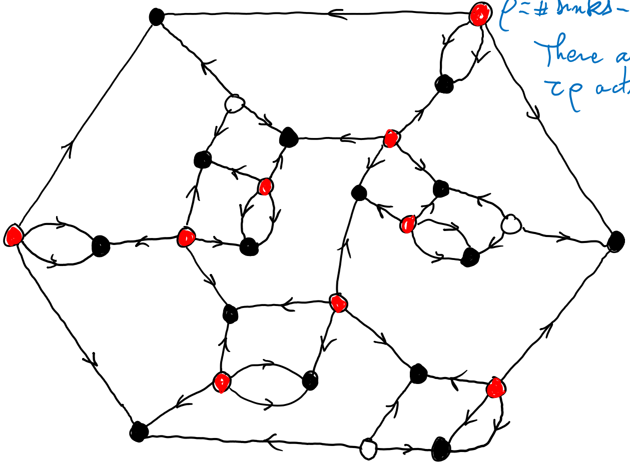
THEOREM
Our reduction of D into a near-regular bipartite digraph B preserves the ρ parameter.

How do reduced digraphs look like?

$\tau = 3$ sources have degree 3 or 4 (active nodes)
sinks have degree 3

ρ = # sinks − # sources

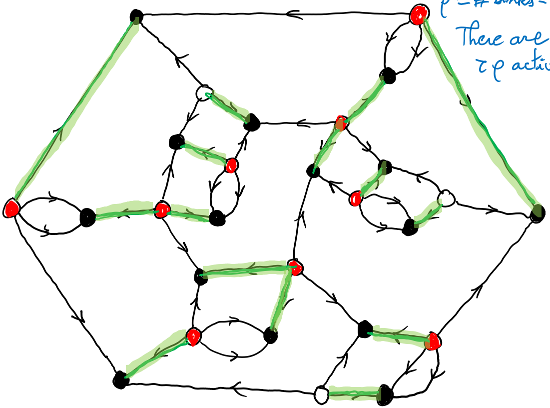
There are $\tau\rho$ active nodes



How do dijoins look like in a packing, if one exists?

$\tau = 3$ sources have degree 3 or 4 (active nodes)
sinks have degree 3

ρ = # sinks − # sources
There are $\tau\rho$ active nodes



$\rho = 3$ of the active nodes have degree 2 in the green subgraph. All other nodes have degree 1.

Theorems (Abdi, Cornuejols, Zlatin)

Consider a digraph D where every node is a source or a sink, all sinks have degree τ, all sources have degree τ or $\tau + 1$ and every dicut has size at least τ.

Let ρ denote the number of sinks of D minus the number of sources of D. Note that $\rho \geq 0$.

THEOREM
Woodall's conjecture is true when ρ equals $0, 1$ or 2.

For $\rho = 0$, this is König's theorem stating that, in any regular bipartite graph, the edges can be partitioned into perfect matchings.

THEOREM
Woodall's conjecture is true when $\rho = 3$ and $\tau = 3$.

Rounded 1-factors

Consider a digraph D where every node is a source or a sink, all sinks have degree τ, all sources have degree τ or $\tau + 1$ and every dicut has size at least τ.

The sources of degree $\tau + 1$ will be called active nodes.

An arc set F of D is a rounded 1-factor if every node of degree τ in D is incident with exactly one arc of F and every active node in D is incident with one or two arcs of F.

THEOREM (de Werra 1971).
The arcs of D can be partitioned into τ rounded 1-factors.

Note that rounded 1-factors are not always dijoins, so de Werra's theorem does not prove Woodall's conjecture.

Using De Werra's theorem, we can show :

THEOREM (Abdi, Cornuejols, Zlatin) If $\rho \leq 1$, there exists an "equitable" packing of τ dijoins.

THEOREM (Abdi, Cornuejols, Zlatin)

For $U \subset V$, let $disc(U)$ denote the number of sinks minus the number of sources in U.

Let J be a rounded 1-factor and let Q denote its nodes of degree 2. Then $|Q \cap U| \geq disc(U)$ for every dicut $\delta^+(U)$.

Furthermore J is a dijoin if, and only if, $|Q \cap U| \geq 1 + disc(U)$ for every dicut $\delta^+(U)$.

REMARK J being a dijoin is solely a function of Q !

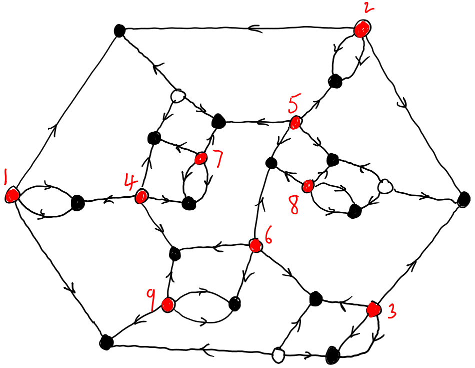
Let $\mathcal{B}$ be the subsets Q of active nodes such that

$$|Q| = disc(V)$$
$$|Q \cap U| \geq 1 + disc(U) \quad \forall \text{ dicut } \delta^+(U) \text{ of } D$$

THEOREM The following statements hold :

1. $\mathcal{B}$ is nonempty (Fujishige 1984).
2. $\mathcal{B}$ is the set of bases of a matroid (Frank and Tardos 1984).

Two matroids



THEOREM

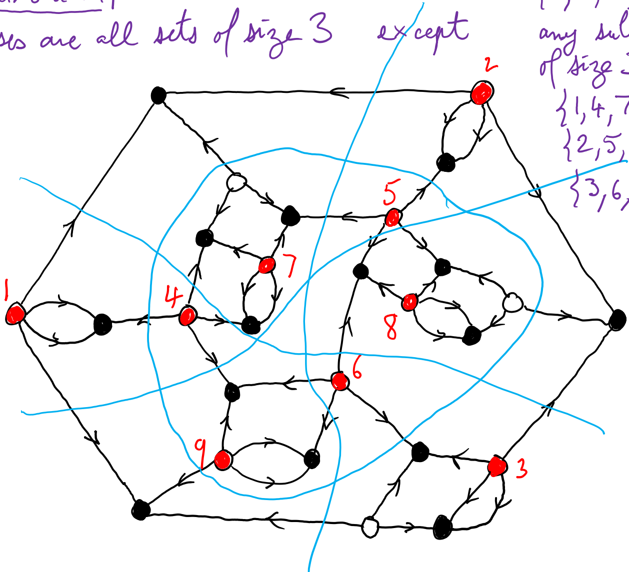
The sets of active nodes that have degree 2 in some rounded 1-factor form the bases of a matroid M_0.

The sets of active nodes that have degree 2 in rounded 1-factors that are dijoins form the bases of a matroid M_1.

Illustration of the matroid M_1

Matroid M_1

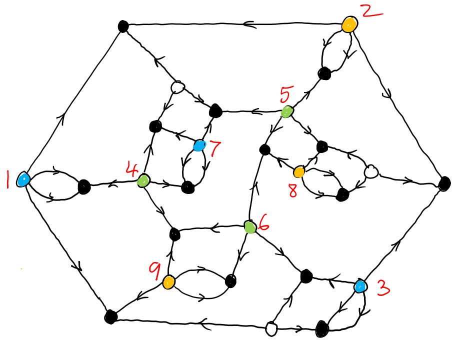
Bases are all sets of size 3 except $\{1,2,3\}$ and any subset of size 3 in
$\{1,4,7,9\}$
$\{2,5,7,8\}$
$\{3,6,8,9\}$



The integer decomposition property

Edmonds (1965) showed that the independent set polytope of a matroid has the integer decomposition property. Namely, for any positive integer k, every integral point in kP can be written as the sum of k integral points in P. As a consequence, we can prove :

THEOREM The active nodes can be partitioned into τ bases of the matroid M_1



Decomposing into a dijoin and a $\tau - 1$ dijoin

- Let $Q_1, \ldots, Q_\tau$ be disjoint bases of M_1.
- Let $b := \chi_{Q_1} + \chi_V$.

Claim 1

There exists a perfect b-matching $x \in \mathbb{Z}_+^E : x(\delta(v)) = b_v \ \forall v \in V$.

PROOF We have

$$\begin{aligned} |Q_1| &= disc(V) \\ |Q_1 \cap U| &\geq disc(U) \quad \forall \text{ dicut } \delta^+(U) \text{ of } D \end{aligned}$$

Equivalently

$$\begin{aligned} b(\text{sources}(V)) &= b(\text{sinks}(V)) \\ b(\text{sources}(U)) &\geq b(\text{sinks}(U)) \quad \forall \text{ dicut } \delta^+(U) \text{ of } D \end{aligned}$$

This is Hall's condition and the claim follows.

Decomposing into a dijoin and a $\tau - 1$ dijoin

- Let $Q_1, \ldots, Q_\tau$ be disjoint bases of M_1.
- Let $b := \chi_{Q_1} + \chi_V$.

Claim 1

There exists a perfect b-matching $x \in \mathbb{Z}_+^A : x(\delta(v)) = b_v$ for all $v \in V$.

Since $b_v = 1$ for every sink v, $x \in \{0, 1\}^E$.

Claim 2

x is the incidence vector of a rounded 1-factor $J_1 \subseteq E$ with dyad centers Q_1. Moreover, J_1 is a dijoin.

This holds because

$$|Q_1| = disc(V)$$
$$|Q_1 \cap U| \geq 1 + disc(U) \quad \forall \text{ dicut } \delta^+(U) \text{ of } D$$

Decomposing into a dijoin and a $\tau - 1$ dijoin

Claim 3
$E - J_1$ is a $\tau - 1$ dijoin.

- Let $Q' := Q_2 \cup \cdots \cup Q_\tau$ and $b' := \chi_{Q'} + (\tau - 1) \cdot \chi_V$.
- χ_{E-J_1} is a perfect b'-matching.
- Let $\delta^+(U)$ be a dicut of D. Then

$$\begin{aligned} |(E - J_1) \cap \delta^+(U)| &= b'(\mathrm{sources}(U)) - b'(\mathrm{sinks}(U)) \\ &= \sum_{i=2}^{\tau} \left(|U \cap Q_i| - disc(U)\right) \\ &\geq \tau - 1 \end{aligned}$$

THEOREM D contains a dijoin and a disjoint $\tau - 1$ dijoin.

Can we make further progress ?

Strongly base orderable matroids

A matroid is strongly base orderable if, for any two bases X, Y, there is a bijection π between $X \setminus Y$ and $Y \setminus X$ such that, for any $S \subset X \setminus Y$, both $X\Delta(S \cup \pi(S))$ and $Y\Delta(S \cup \pi(S))$ are bases.

THEOREM The matroid M_0 is strongly base orderable.

THEOREM When the matroid M_1 is strongly base orderable, D contains τ disjoint dijoins.

But, this is not always the case. Our proof of Woodall's conjecture for $\rho = 3$ and $\tau = 3$ uses the fact that the only matroid on 6 elements that is not strongly base orderable is the cycle matroid of K_4 (Brualdi 1971).

Denote by K_4 the complete graph on 4 vertices and by $M(K_4)$ the cycle matroid of K_4.

LEMMA Brualdi 1971 Up to isomorphism, $M(K_4)$ is the only matroid on at most six elements that is not strongly base orderable.

Using matroid machinery, we can prove :

LEMMA Let M be a matroid over 9 elements whose ground set can be partitioned into bases Q_1, Q_2, Q_3. Then we may choose Q_1, Q_2, Q_3 such that $M|(Q_i \cup Q_j) \ncong M(K_4)$ for some distinct $i, j \in \{1, 2, 3\}$.

THEOREM The arc set of a sink-regular $(3, 4)$-bipartite digraph such that $\rho \leq 3$ can be partitioned into 3 disjoint dijoins.

THEOREM Let D be a digraph where every dicut has size at least 3. Suppose $\rho \leq 3$. Then there exist 3 disjoint dijoins.

THEOREM Let $\tau \geq 3$ be an integer, and D a sink-regular $(\tau, \tau+1)$-bipartite digraph such that $\rho = 3$. There exist disjoint bases $Q_1, \dots, Q_\tau$ of M_1 such that $M_1|(Q_1 \cup Q_2)$ is strongly base orderable.

PROOF There exist disjoint bases $Q_1, \dots, Q_\tau$ of M_1. Consider the matroid $M := M_1|(Q_1 \cup Q_2 \cup Q_3)$, which has 9 elements and its ground set is partitioned into bases Q_1, Q_2, Q_3. By the previous lemma, we may choose Q_1, Q_2, Q_3 such that $M|(Q_1 \cup Q_2) \not\cong M(K_4)$, so by Brualdi's lemma, $M|(Q_1 \cup Q_2)$ is strongly base orderable. Since $M_1|(Q_1 \cup Q_2) = M|(Q_1 \cup Q_2)$, the disjoint bases $Q_1, Q_2, Q_3, Q_4, \dots, Q_\tau$ prove the theorem.