Lecture 3: Total Dual Dyadicness

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Dyadic Linear Programming

A rational number is dyadic if it is an integer multiple of $\frac{1}{2^k}$ for some nonnegative integer k.

Dyadic numbers can be represented exactly on a computer in binary floating-point arithmetic.
Therefore they are important for numerical computations.

$$DLP \qquad \begin{array}{c} \sup w^\top x \\ Ax \leq b \\ x \text{ dyadic} \end{array}$$

where a vector x is dyadic if each of its components is a dyadic rational.

If we require dyadic numbers with k bounded by a fixed given K, (DLP) reduces to integer programming.

Our Motivation

Let A be an $m \times n$ matrix with $0-1$ entries. Consider the primal-dual pair of linear programs for $w \in \mathbb{Z}^n$:

$$\min\left\{w^\top x : Ax \geq \mathbf{1}, x \geq 0\right\} \quad = \quad \max\left\{\mathbf{1}^\top y : A^\top y \leq w, y \geq 0\right\}.$$

If the primal has an integral optimal solution for every $w \in \mathbb{Z}^n$, then A is called an *ideal matrix*.

Seymour made the following intriguing conjecture.

THE DYADIC CONJECTURE Seymour 1975
Let A be an ideal $m \times n$ matrix. Then for every $w \in \mathbb{Z}^n$, the dual has an optimal solution that is dyadic.

In fact, Seymour 1975 asks whether a $1/4$-integral solution might be possible (recall Lecture 1).

Our Motivation (continued)

For a 0,1 matrix A, consider the primal-dual pair

$$\min\left\{w^\top x : Ax \geq \mathbf{1}, x \geq 0\right\} \quad = \quad \max\left\{\mathbf{1}^\top y : A^\top y \leq w, y \geq 0\right\}.$$

THE DYADIC CONJECTURE Seymour 1975
Let A be an ideal $m \times n$ matrix. Then for every $w \in \mathbb{Z}^n$, the dual has an optimal solution that is dyadic.

A STRONGER CONJECTURE Seymour 1975
If A is ideal, the dual has a $1/4$-integral solution.

THEOREM Abdi, Cornuéjols, Guenin, Tuncel (SIDMA 2021)
The dyadic conjecture is true if the optimal value is 2.

THEOREM Abdi, Cornuéjols, Palion 2021
The dyadic conjecture is true for T-joins.

Some questions

$$DLP \qquad \begin{array}{c} \sup\ w^\top x \\ Ax \leq b \\ x \text{ dyadic} \end{array}$$

- When is this problem feasible ?
- When does it have an optimum solution ?
- Can feasibility be checked in polynomial time ?
- Can dyadic linear programs be solved in polynomial time ?
- When can we guarantee that the dual also has a dyadic optimal solution ?

Dyadic solutions to linear programs

LEMMA

A nonempty rational polyhedron contains a dyadic point if and only if its affine hull contains a dyadic point.

LEMMA

Consider a linear system $Ax = b$, where A, b have integral entries. Then exactly one of the following statements holds :

1. $Ax = b$ has a dyadic solution,
2. there exists a vector $y \in \mathbb{R}^m$ such that $y^\top A$ is integral and $y^\top b$ is non-dyadic.

THEOREM

Let P be a nonempty rational polyhedron whose affine hull is $\{x : Ax = b\}$. Exactly one of the following statements holds.

- P contains a dyadic point,
- there exists a y such that $y^\top A$ is integral and $y^\top b$ is non-dyadic.

Optimization

$$DLP \qquad \begin{array}{c} \sup\ w^\top x \\ Ax \leq b \\ x \text{ dyadic} \end{array}$$

THEOREM The status of (DLP) can be classified as follows :

1. (DLP) is infeasible,
2. (DLP) is unbounded,
3. (DLP) has an optimal solution,
4. (DLP) is not unbounded, has feasible solution(s) and a finite optimal value, but no optimal solution.

Moreover, in cases 3 and 4, the value of the supremum in (DLP) is the max objective value of the underlying LP.

THEOREM
Dyadic linear programs can be solved in polynomial time.

Dyadic poyhedra

Let $P = \{x : Ax \leq b\}$ be a polyhedron where A, b are integral.

THEOREM The following are equivalent :

- Every nonempty face of P contains a dyadic point.
- For every nonempty face F, $\mathrm{aff}(F)$ contains a dyadic point.
- $\forall w \in \mathbb{R}^n$ s.t. $\max\{w^\top x : Ax \leq b\}$ has a finite optimum, it has a dyadic optimal solution.
- $\forall w \in \mathbb{Z}^n$ s.t. $\max\{w^\top x : Ax \leq b\}$ has a finite optimum, it has a dyadic optimal value.

Note that the second statement is novel, compared to the integral case.

DEFINITION A polyhedron is dyadic if any of these equivalent conditions hold.

QUESTION What is the complexity of testing that a polyhedron is dyadic ? Clearly in coNP .

Totally dual dyadic systems

DEFINITION

$Ax \leq b$ is totally dual dyadic if, for all $w \in \mathbb{Z}^n$ for which $\min\left\{b^\top y : A^\top y = w, y \geq 0\right\}$ has a solution, it has a dyadic optimal solution.

COROLLARY If $Ax \leq b$ is totally dual dyadic, then $\{x : Ax \leq b\}$ is a dyadic polyhedron.

THEOREM The following are equivalent :

- $Ax \leq b$ is totally dual dyadic.
- For every nonempty face F the tight rows of A form a dyadic generating set for the conic hull. (the dyadic analogue of a Hilbert basis)
- For every nonempty face F the tight rows of A form a dyadic generating set for the span.

Note that the third statement is novel, compared to the integral case.

Dyadic generating sets for subspaces

DEFINITION $\{a^1, \ldots, a^m\}$ is a dyadic generating set for the span if every integral vector in the span of these m vectors can be expressed as a dyadic linear combination of $\{a^1, \ldots, a^m\}$.

THEOREM Let A be an integral matrix. The following are equivalent :

- the rows of A form a dyadic generating set for the span.
- the columns of A form a dyadic generating set for the span.
- whenever $y^\top A$ and Ax are integral, $y^\top Ax$ is dyadic.
- every elementary divisor of A is a power of 2.
- the GCD of the subdeterminants of A of order $\operatorname{rank}(A)$ is a power of 2.
- there exists a dyadic matrix B such that $ABA = A$

Totally dual dyadic systems

THEOREM One can check in polynomial time whether the rows of an integral matrix form a dyadic generating set for the span.

RECALL THE DEFINITION

$Ax \leq b$ is totally dual dyadic if, for all $w \in \mathbb{Z}^n$ for which $\min\left\{b^\top y : A^\top y = w, y \geq 0\right\}$ has a solution, it has a dyadic optimal solution.

COROLLARY

Testing total dual dyadicness belongs to coNP.

Question

What is the complexity of testing total dual dyadicness ?

COROLLARY

If every subdeterminant of A is 0 or $\pm$ a power of 2 , then $Ax \leq b$ is totally dual dyadic.

Cycle double covers

Let $G = (V, E)$ be a graph.
Let A be the $0, 1$ matrix whose rows correspond to E and whose columns are the incidence vectors of the cycles of G.

THE CYCLE DOUBLE COVER CONJECTURE
Szekeres 1973, Seymour 1981
$Ax = \mathbf{1}, x \geq 0$ has a $1/2$-integral solution.

THEOREM Goddyn 1993
$Ax = \mathbf{1}$ has a $1/2$-integral solution.

COROLLARY
$Ax = \mathbf{1}, x \geq 0$ has a dyadic solution.

QUESTION
Can we guarantee a small denominator ?

Perfect matchings

Let $G = (V, E)$ be an r-graph. That is G is an r-regular graph on an even number of vertices, and $|\delta(U)| \geq r$ for all odd cardinality $U \subseteq V$.

Let A be the $0, 1$ matrix whose rows correspond to E and whose columns are the incidence vectors of the perfect matchings of G.

THE GENERALIZED BERGE-FULKERSON CONJECTURE
Seymour 1979
$Ax = \mathbf{1}, x \geq 0$ has a $1/2$-integral solution.

THEOREM Seymour 1979 $r = 3$, Lovász 1987 $r \geq 4$
$Ax = \mathbf{1}$ has a $1/2$-integral solution.

COROLLARY
$Ax = \mathbf{1}, x \geq 0$ has a dyadic solution.

QUESTION
Can we guarantee a small denominator ?

The dyadic conjecture

Let A be an $m \times n$ matrix with $0, 1$ entries.

For all $w \in \mathbb{Z}_+^n$, consider
$\min\left\{w^\top x : Ax \geq \mathbf{1}, x \geq 0\right\} = \max\left\{\mathbf{1}^\top y : A^\top y \leq w, y \geq 0\right\}$.

If the primal has an integral optimal solution for all $w \in \mathbb{Z}_+^n$, the matrix A is called ideal.

THE DYADIC CONJECTURE   Seymour 1975
Let A be an ideal $0, 1$ matrix. Then for all $w \in \mathbb{Z}_+^n$, the dual has an optimal solution that is dyadic.

THEOREM   Abdi, Cornuéjols, Guenin, Tuncel   SIDMA 2021
The dyadic conjecture is true if the optimal value is 2.

In fact, we prove this theorem for 0,1 matrices without delta and extended odd hole minors (a class that contains ideal matrices).

THEOREM   Abdi, Cornuéjols, Palion   2021
The dyadic conjecture is true for T-joins.

Proof of the dyadic conjecture for T-joins

The proof uses two ingredients.

THEOREM Lovász 1987
Let $G = (V, E)$ be an r-graph for some integer $r \geq 2$.
If $w \in \mathbb{R}^{|E|}$ is matching-integral, then $\mathbf{1}^\top w$ is $1/2$-integral.

THEOREM Abdi, Cornuéjols, Guenin, Tuncel 2021
Let P be a nonempty rational polyhedron whose affine hull is $\{x : Ax = b\}$. Exactly one of the following statements holds.

- P contains a dyadic point,
- there exists a y such that $y^\top A$ is integral and $y^\top b$ is non-dyadic.

Proof of the dyadic conjecture for T-joins

THEOREM Abdi, Cornuéjols, Palion 2021
Take an integer $r \geq 2$, and let $G = (V, E)$ be an r-graph where every minimum odd cut is trivial. Then there is an assignment of a nonnegative dyadic rational y_M to every perfect matching M such that $\mathbf{1}^\top y = r$, and $\sum (y_M : M \ni e) = 1$ for each $e \in E$.

PROOF Consider the following primal-dual pair :

$$\begin{array}{llr} \min & \mathbf{1}^\top x & \\ \text{s.t.} & \sum (x_e : e \in J) \geq 1 & \forall J \text{ a } V\text{-join} \\ & x_e \geq 0 & \forall e \in E \end{array} \quad = \quad \begin{array}{llr} \max & \mathbf{1}^\top y & \\ \text{s.t.} & \sum (y_J : e \in J) \leq 1 & \forall e \in E \\ & y_J \geq 0 & \forall J \text{ a } V\text{-join.} \end{array}$$

By our hypothesis, the optimal primal solutions are precisely the trivial cuts of G. The optimal value is r.
By strict complementary slackness,

($\star$) *there exists an optimal dual solution that is nonzero on every perfect matching.*

Proof of the dyadic conjecture for T-joins

Now let A be the matrix whose rows are indexed by the edges, and whose columns are the incidence vectors of the perfect matchings of G.

The optimal dual solutions are in a one-to-one correspondence with the vectors in the polyhedron $P := \{y : Ay = \mathbf{1}, y \geq 0\}$.

By $(\star)$, P contains a vector that is nonzero on every coordinate, implying in turn that the affine hull of P is $\{y : Ay = \mathbf{1}\}$.

By Lovász' theorem, if $w \in \mathbb{R}^{|E|}$ is matching-integral, then $\mathbf{1}^\top w$ is $1/2$-integral. That is, the second statement in our theorem of the alternative does not hold. Therefore the first statement holds, namely P contains a dyadic vector, as required.

Proof of the dyadic conjecture for T-joins

Let $G = (V, E)$ be a graph, and $T \subseteq V$ a nonempty set of even cardinality. Let τ be the minimum cardinality of a T-cut. We need to prove that there exists a dyadic fractional packing of T-joins of value τ. We proceed by induction on the size of the graph.

We may assume that every edge belongs to a minimum T-cut and that every such T-cut is trivial.

Now double every edge of G to get an Eulerian graph H, where the minimum cardinality of a T-cut is 2τ.
Let $T' := \{v \in T : \deg_H(v) = 2\tau\}$.
Observe that every vertex of H is either in T' or is adjacent to only vertices in T'. In fact $T' = V$ (otherwise we could reduce the size of the graph). By the theorem above, H has a dyadic fractional packing of V-joins of value 2τ.
Going back to G, every edge has congestion 2. Observe now that $\frac{1}{2}y$ is a dyadic fractional packing of T-joins of G of value τ, as desired.

Proof of the dyadic conjecture for T-joins

Note that dyadic denominators show up in three places.

First in Lovasz's result on matching-integral vectors,

second in the application of our theorem of the alternative,

and third in the final stage where edges are doubled in order to make the graph Eulerian.

Future Work and Research Directions

1. Does Seymour's dyadic conjecture hold for dijoins ?