Proof of the Goldberg-Seymour Conjecture – I

11th Cargèse Workshop on Combinatorial Optimization

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Graphs and simple graphs

Graphs: mulitigraphs – a finite graph which is permitted to have multiple edges, but no loops.

Simple graphs: No multiple edges and no loops

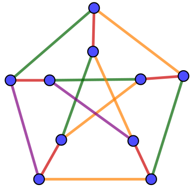
Edge coloring - basic

Edge coloring: an assignment of "colors" to the edges of the graph so that no two adjacent edges have the same color
a color class = matching

Edge k-coloring: an edge coloring using k colors from the palette $[k]$

Chromatic index $\chi'(G)$: smallest integer k so that G has an edge k-coloring

Edge-coloring problem (ECP): find an edge coloring of a graph G with $\chi'(G)$.



Hardness of ECP

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- There is no efficient algorithm for solving ECP exactly unless $NP = P$.
- The focus of extensive research has been on near-optimal solutions to ECP or good estimates of $\chi'(G)$.

classical bounds

A trivial lower bound:

$\chi'(G) \geq \Delta(G)$

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Vizing, 1964 and Gupta, 1966:

$\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum multiplicity of edges in G.

simple graphs

Vizing's theorem for simple graphs:

If G is a simple graph, then $\Delta(G) \leq \chi'(G) \leq \chi'(G) + 1$.

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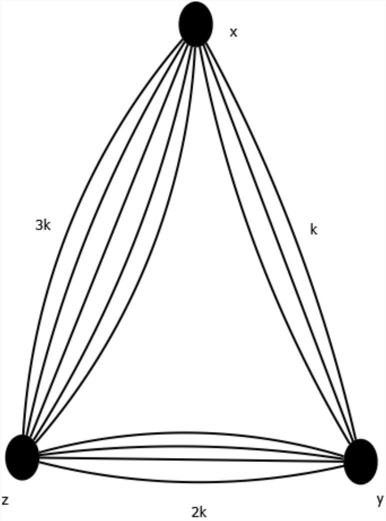
Central theme of ECP for simple graphs:

the classification problem

gaps for multigraphs



- $\chi'(G) = |E(G)| = 6k$

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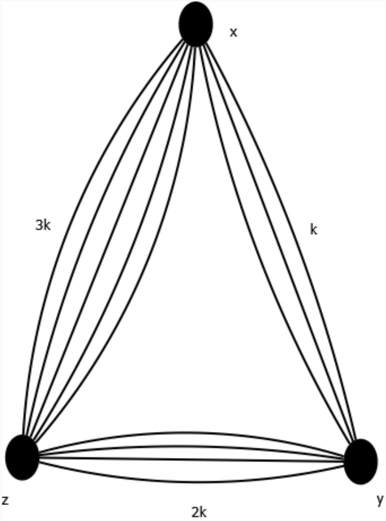

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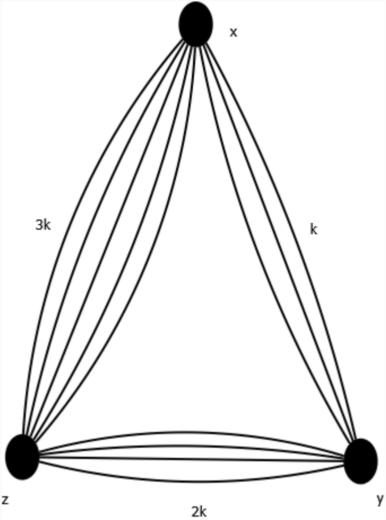

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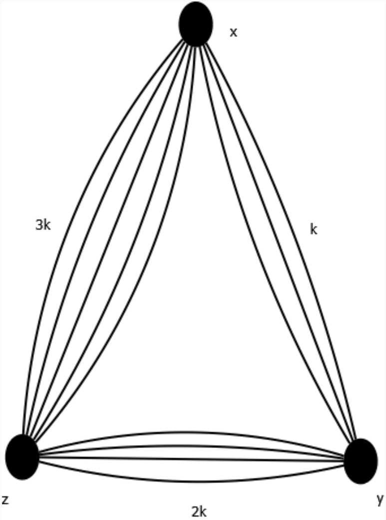

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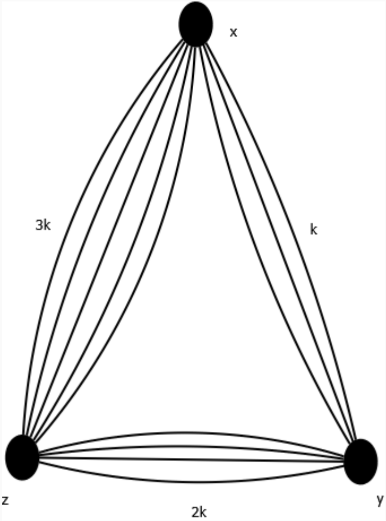

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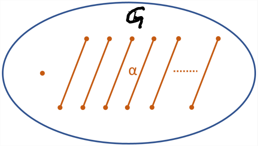
- $3\Delta(G)/2 = 7.5k.$

another lower bound

Let $\overline{\varphi}$ be a k-edge-coloring of a graph G and E_α be the set of edges colored by color α.

- $|E_\alpha| \leq \lfloor \frac{|G|}{2} \rfloor$, where $|G| = |V(G)|$.



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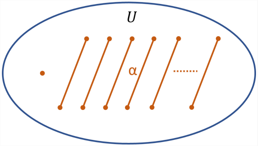

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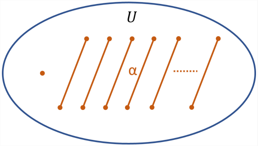

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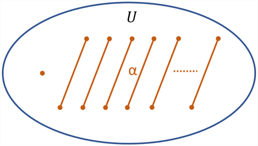

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- The above inequality indeed holds for every induced subgraph H of G.
- $k \geq \max\left\{ \frac{|E(H)|}{\lfloor |H|/2 \rfloor} : \ H \subseteq G, \ |H| \geq 3 \right\}$.

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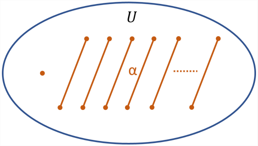

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- The above inequality indeed holds for every induced subgraph H of G.
- $k \ge \max\left\{ \frac{|E(H)|}{\lfloor |H|/2 \rfloor} : \; H \subseteq G, \; |H| \ge 3 \right\}$.
- When $|H|$ is even, we have $\frac{|E(H)|}{\lfloor |H|/2 \rfloor} \le \Delta(H) \le \Delta(G)$.

density

fractional density

$$\omega^*(G) := \max\left\{\frac{2|E(H)|}{|H|-1} :\ H \subseteq G,\ |H| \geq 3 \text{ and odd}\right\}$$

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How good is this lower bound?

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- $\mu(|G| - 1) < \Delta$, a contradiction.

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Goldberg, 1973 and Seymour, 1974

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There are only two possibilities for $\chi'(G)$:
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More closer form:

There are only two possibilities for $\chi'(G)$:
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Recall $\omega(G) = \lceil \omega^*(G) \rceil$.

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Chen, Zang and Zhao, 2019:

a combinatorial polynomial-time algorithm for finding the density $\omega^*(G)$.

Approximate results

In the form

$$\chi'(G) \leq \max\{\Delta(G) + \rho(G), \lceil \omega(G) \rceil\},$$

where $\rho(G)$ is a positive number depending on G.

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- $\rho(G) \leq \sqrt[3]{\Delta(G)/4}$ (C., and Jing, 2019)

an equivalent conjecture

A stronger version of Jakobsen's conjecture, 1975

$\chi'(G) \leq \frac{m}{m-1} \cdot \Delta(G) + \frac{m-3}{m-1}$ for every odd integer $m \geq 3$.

- $m = 5$ (Andersen, 1977; Goldberg, 1973)

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random graphs

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- if n is even and $m := m(n)$, then $\chi'(M) = \Delta(M)$ for a typical $M \sim M(n, m)$.
- for fixed $\epsilon > 0$, if n is odd, then a typical $M \sim M(n, m)$ has $\chi'(M) = \Delta(M)$ for $m \leq (1 - \epsilon)n^3$, and $\chi'(M) = \omega(M)$ for $m \geq (1 - \epsilon)n^3$.

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If G is an r-graph, then $\omega(G) \leq r$, and so
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r-graphs

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an r-regular (multi-)graph such that
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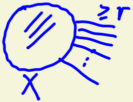

the main theorem

C., Jing, and Zang, 2019+

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what is this theorem?

If $\chi'(G) = k + 1 \geq \Delta(G) + 2$, then G has a subgraph H with odd number of vertices such that
for every edge $e \in E(H)$, edges $H - e$ can be decomposed into k near-perfect matchings.

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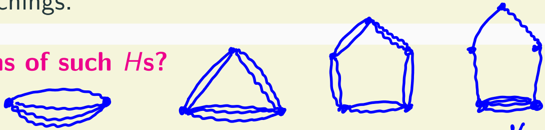
characterizations of such Hs?

$|H| \leq 3$



✓ ✗

some ring graphs –cycles with multiple edges

others???

graph edge covering

edge cover F:

an edge set F such that each vertex of G is incident to at least one edge in F, i.e.,

an edge vector x such that $Ax \geq 1$ (Cornuéjois)

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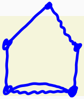
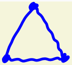


$\xi = 2$



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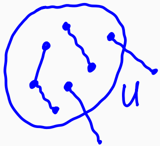
difficulty:

determining the cover index $\xi(G)$ is NP-hard.

lower bounds

For each $U \subseteq V(G)$, let $E^+(U)$ be the set of all edges of G with at least one end in U for each $U \subseteq V$, i.e., $|E^+(U)| = |E(U)| + |\partial(U)|$.



Needs $\geq \frac{|U|+1}{2}$ edges to cover U.

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fractional co-density and co-density:

$\Phi^*(G) = \min\left\{\frac{2|E^+(U)|}{|U|+1} : \ U \subseteq V, \ |U| \geq 3 \ \text{and} \ \text{odd}\right\}.$
$\Phi(G) = \lfloor \Phi^*(G) \rfloor.$

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upper bound:

$\xi(G) \leq \min\{\delta(G), \Phi(G)\}.$

$\updownarrow$

recall:

$\chi'(G) \geq \min\{\Delta(G), \omega(G)\}.$

fractional cover index

fractional cover index $\xi^*(G)$:

the optimal value of the *fractional edge cover packing problem* (FECPP):

$$\begin{array}{ll} \text{Maximize} & \mathbf{1}^T \boldsymbol{x} \\ \text{subject to} & B\boldsymbol{x} = \mathbf{1} \\ & \quad \boldsymbol{x} \geq \mathbf{0}, \end{array}$$

where B is the edge-edge cover incidence matrix of G.

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where B is the edge-edge cover incidence matrix of G.

Zhao, Chen, and Sang, 2020:

$\xi^*(G) = \min\{\delta(G), \Phi^*(G)\}$.
Moreover, there is a combinatorial polynomial-time algorithm for finding the fractional co-density $\Phi^*(G)$ of any multigraph G.

Gupta's cover index conjecture

Gupta, 1978:

$\xi(G) \geq \min\{\delta(G) - 1, \Phi(G)\}$.

the Goldberg-Seymour conjecture:

$\chi'(G) \leq \max\{\Delta(G) + 1, \omega(G)\}$.

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Gupta, 1978:

$\xi(G) \geq \min\{\delta(G) - 1, \Phi(G)\}.$

Gupta, 1978:

$\xi(G) \geq \delta(G) - \mu(G)).$

$\xi(G) \geq \min\{\lfloor \frac{7\delta(G)+1}{8} \rfloor, \lfloor \Phi(G) \rfloor\}.$

$\delta - \frac{1}{8}\delta + \frac{1}{8}$

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Cao, C., Ding, Jing, and Zang, 2020+:

$\xi(G) \geq \min\{\delta(G) - 1, \Phi(G)\}$ if $\Phi(G)$ is not integral and

$\xi(G) \geq \min\{\delta(G) - 2, \Phi(G) - 1\}$ otherwise.

proof: We may assume that G is $\theta^+(G) =: k$ "regular".

$$\theta^+(G) = k \xrightarrow{\text{G-S conj.}} \chi'(G) \le k+3.$$

let $\underbrace{M_1, M_2, \dots, M_{k-1}}, \underbrace{M_k, M_{k+1}, M_{k+2}, M_{k+3}}$

be an edge-coloring of G.

let $H = M_k \cup M_{k+1} \cup M_{k+2} \cup M_{k+3}$. and

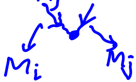
N be an orientation of H. s.t

$$|d_N^+(v) - d_N^-(v)| \le 1$$

$\forall\ v \in V(G)$.

Case 1. $\exists 1$ i, s.t. $d_{M_i}(v) = 0.$ $\longrightarrow d_N^-(v) \ge 1$

Case 2: $\exists 2$ i, j. s.t $d_{M_i}(v) = d_{M_j}(v) = 0$ $\longrightarrow d_N^-(v) \ge 2$



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an assignment of colors to the edges and vertices of G such that no two adjacent edges receive the same color, no two adjacent vertices receive the same color and no edge has the same color as its two endpoints.

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Behzad's total-coloring conjecture, 1965:

$\chi''(G) \leq \Delta(G) + 2$ for every simple graph G.

special graph H

special graph H:

$\chi'(H) = k + 1 \geq \Delta(H) + 2$, $H - e$ is a disjoint union of k near-perfect matchings for any edge $e \in E(H)$, say $M_1, \ldots, M_k$, which gives a $(k+1)$-edge-coloring of H

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a total coloring of H:

for each vertex v, since $k \geq \Delta(H) + 1$, there is a color i not used by edges incident to v, assign i to v.

χ' vs χ''

Goldberg's total-coloring conjecture, 1984:

If $\chi'(G) \geq \Delta(G) + 3$, then $\chi''(G) = \chi'(G)$.

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C., and Hao, 2021+:

If $\chi'(G) \geq \Delta(G) + 10\Delta(G)^{35/36}$, then $\chi''(G) = \chi'(G)$ provided $\Delta(G)$ sufficiently large. So $\cdot\ \chi' \geq \Delta + o(\Delta)$

more applications

- characterization when $\chi'(G) = \Delta(G) + \mu(G)$ when $\mu(G) \geq 2$ (Cao, C., He, Jing, 2020)

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- characterization when $\chi'(G) = \Delta(G) + \mu(G)$ when $\mu(G) \geq 2$ (Cao, C., He, Jing, 2020)
- precoloring extension (Cao, C. Jing, Qi, Shan, 2021+)

Classification of (multi)-graphs

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A new classification:

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- If G is a critical simple graph with $\delta(G_\Delta) \leq k$ and $\Delta(G) \geq \frac{2n}{3} + \frac{3k}{2}$, then G is of the first class.

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Cao, C., Jing, Shan, 2021+:

If $\Delta(G) - 7\delta(G)/4 \geq (3n - 17)/4$, then G is of the first class. So for $0 < \epsilon < 1/7$, if $\delta(G) \leq \epsilon n$ and $\Delta \geq (3n - 17 + 7\epsilon n)/4$, then G is of the first class.

double edges

the Berge-Fulkerson conjecture, 1971:

if G is a bridgeless cubic graph, then G contains six perfect matchings such that each edge is in exactly two of them, which is equivalent to saying that $\chi'(2G) = 6$

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double graph conjecture:

$2G$ is of the first class for any graph G.

planar graphs

Seymour's exact conjecture, 1979

All planar graphs are of the first class.


Thank You !