Proof of the Goldberg-Seymour Conjecture - II

11th Cargèse Workshop on Combinatorial Optimization

Guantao Chen

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Overview

General setting

the theorem

the theorem:

If $\chi'(G) \geq \Delta(G) + 2$ then $\chi'(G) = \omega(G)$,
where

$$\begin{aligned} \omega(G) &= \lceil \omega^*(G) \rceil \\ \omega^*(G) &= \max \left\{ \frac{2|E(H)|}{|H| - 1} : \ H \subseteq G, \ |H| \geq 3 \text{ and odd} \right\} \end{aligned}$$

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Assuming:

G is a critical graph, $\chi'(G) = k + 1 \geq \Delta(G) + 2$, $e \in E(G)$ and $\varphi \in \mathcal{C}^k(G - e)$ is a k-edge coloring of $G - e$.

some basic notation on coloring

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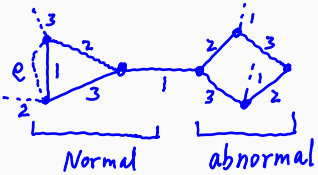
- $\varphi(v) = \varphi(\partial(v))$, the set of colors present at v
  $\overline{\varphi}(v) = [k] - \varphi(v)$, the set of colors missing at v.
- For each $X \subseteq V(G)$, define $\overline{\varphi}(X) = \cup_{x \in X} \overline{\varphi}(x)$.
  subgraph $H \subseteq G$, $\overline{\Phi}(H) := \overline{\Phi}(V(H))$

normal, closed, and strongly closed

$U \subseteq V$ **normal:** (borrowed from Andras)

$\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for any two distinct vertices $u, v \in U$.



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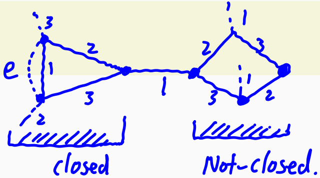
U does not have lobes.



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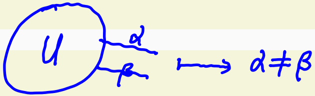

closed and colors on $\partial(U)$ are distinct, i.e., $|\partial_\alpha(U)| \leq 1$ for each color α.

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$E_\alpha(U)$ is a near-perfect matching for $G[U]$ for every color $\alpha \in [k]$.

$\rightsquigarrow |E_\alpha(U)| = \frac{|U|-1}{2}$

$\rightsquigarrow |E(U)| = \sum_{\alpha \in [k]} |E_\alpha[U]| + |\{e\}|$

$= \frac{k(|U|-1)}{2} + 1$

$\rightsquigarrow \frac{2|E(U)|}{|U|-1} = k + \dots > k$

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$\omega^*(G) > k$**:**

$|E[U]| = k(|U| - 1)/2 + 1.$

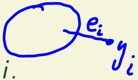
Tashkinov tree

Tashkinov tree

Definition

A Tashkinov tree with respect to e and φ is a sequence $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with $p \geq 1$:

(T1) *The vertices $y_0, \ldots, y_p$ are distinct, $e_1 = e$ and and for $i = 1, \ldots, p$, we have $e_i \in E_G(\{y_0, \ldots, y_{i-1}\}, y_i)$.*



(T2) *For every $i \geq 2$, $\varphi(e_i) \in \overline{\varphi}(y_h)$ for some $h < i$.*

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The definition of Tashkinov tree also give a linear order of its vertices and edges.

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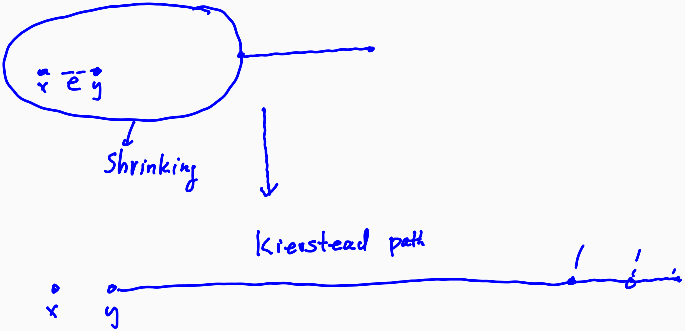
Theorem (Tashkinov, 2000)

$V(T)$ is elementary provided $\chi'(G) = k + 1 \geq \Delta(G) + 2$ and e is a critical edge, and $\varphi \in \mathcal{C}^k(G - e)$.

proof of Tashkinov theorem



* The path may be much longer that $|V(T)|$.

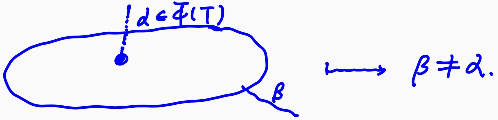
maximal Tashkinov trees – closure:

Observation:

all maximal Tashkinov trees are closed (its vertex set) and have the same set of vertices.

↓ no lobes

also called a closure.



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problem:

Can we find a way to extend a Tashkinov tree to a strongly closed set, but keep the normality?

too many missing colors

$|X| \leq k/(k - \Delta)$:

for every elementary set X such that $e \in E[X]$.

normal

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Proof.

Otherwise, since $|\overline{\varphi}(v)| \geq k - \Delta$ and ..., we have
$k \geq \sum_{v \in X} |\overline{\varphi}(x)| + 2 > |X|(k - \Delta) > k$, a contradiction. □

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Scheide, 2010, C., Yu and Zang, 2009:

If $\chi'(G) \geq \Delta + \sqrt{\Delta/2}$, then $\chi'(G) = \omega(G)$.

Scheide's proof

Proof.

- Suppose there does not exist a strongly closed maximal Tashkinov tree.

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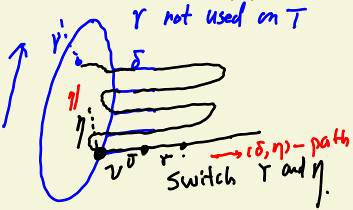

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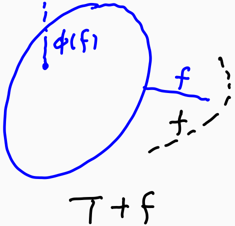
- $|V(T)| \geq 2(k - \Delta) + 2$
- $|\overline{\varphi}(V(T)| = \sum_{v \in V(T)} 2|\overline{\varphi}(v)| > 2|V(T)| \cdot (k - \Delta) > k$, a contradiction.

□

an extension

TAA algorithm:

Suppose we have a tree sequence $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ and $f \in \partial(T)$. If $\varphi(f) \in \overline{\varphi}(T)$, let $T := T + f = (y_0, e_1, y_1, \ldots, e_p, y_p, f, y_{p+1})$, where y_{p+1} is the end of f outside T.



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starting with $V(e)$, apply TAAs repeadily.

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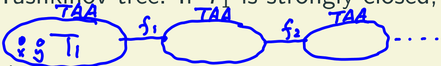

- find an edge $f_1 \in \partial(T_1)$ and add f to T, and apply TAA to $T + f_1$ to get a closed tree sequence T_2. If T_2 is strongly closed, stop; other wise

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- $T_3 \ldots$

bounded by $|\overline{\varphi}(T_1)|$

C., Gao, Kim, Postle, and Shan, 2019:

Suppose $\chi'(G) \neq \omega(G)$. For every extended Tashkinov tree T we have

$$|E(T)| > 2|\overline{\varphi}(T_1)|.$$

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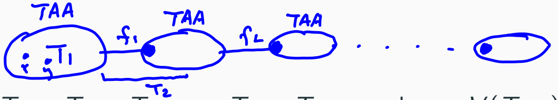
- $|\overline{\varphi}(T)| > |T|(k - \Delta)^2 \geq 2(k - \Delta)^3$.

□

the first vertices



Let $T_1 \subset T_2 \subset T_3 \cdots \subset T_n \subset T_{n+1}$, and $v_i \in V(T_{i+1})$ is the first vertex added to T_i.

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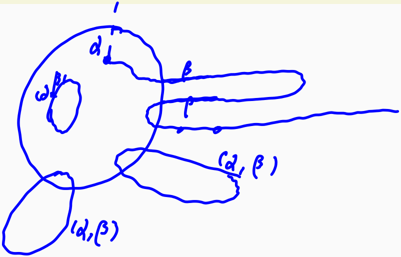
- We CANNOT prove $T_i + v_i$ is elementary in general.
- We cannot restrict our consideration to one coloring φ or a few colorings related to it.

the structure of extension

a property of maximal Tashkinov tree

Lemma

Let (G, e, φ) be a k-triple, let T be a maximal Tashkinov tree with respect to e and φ, and let α and β be two colors in $[k]$ with $\overline{\varphi}(T) \cap \{\alpha, \beta\} \neq \emptyset$. Then there is at most one (α, β)-path with respect to φ intersecting T.



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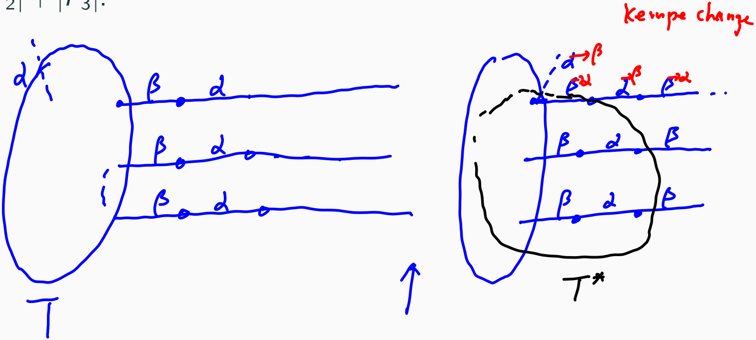
proof

Assume the contrary: there are at least two (α, β)-paths Q_1 and Q_2 with respect to φ intersecting T. Since $V(T)$ is normal and closed, precisely one of α and β, say α, is in $\overline{\varphi}(T)$. Thus at least three ends of Q_1 and Q_2 are outside T. Traversing Q_1 and Q_2 from these ends respectively, we can find at least three $(T, \varphi, \{\alpha, \beta\})$-exit paths P_1, P_2, P_3. We call the tuple $(\varphi, T, \alpha, \beta, P_1, P_2, P_3)$ a *counterexample* and use $\mathcal{K}$ to denote the set of all such counterexamples.

let $(\varphi, T, \alpha, \beta, P_1, P_2, P_3)$ be a counterexample in $\mathcal{K}$ with the minimum $|P_1| + |P_2| + |P_3|$.



colorings?

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- Normal w.r.t. which coloring?
- We cannot say for all colorings in $\mathcal{C}^k(G-e)$
- Special colorings?

stable coloring

a (T, C, φ)-stable coloring π:

A coloring satisfying the following two conditions.

(i) $\pi(f) = \varphi(f)$ for any $f \in E$ incident to T with $\varphi(f) \in \overline{\varphi}(T) \cup C$; and

(ii) $\overline{\pi}(v) = \overline{\varphi}(v)$ for any $v \in V(T)$.

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Being $(T, C, \cdot)$-stable is an equivalence relation on $\mathcal{C}^k(G - e)$.

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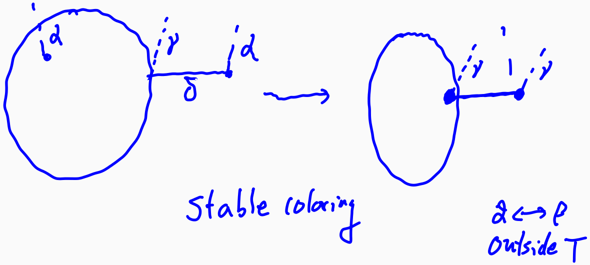
Lemma

Suppose T is closed but not strongly closed with respect to φ, with $|V(T)|$ odd. If π is a (T, C, φ)-stable coloring, then T is also closed but not strongly closed with respect to π.

exit path

exit path of T:

a path intersects T at exactly one vertex.



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Lemma

Suppose T is closed with respect to φ, and $f \in E(u, v)$ is an edge in $\partial(T)$ with $v \in V(T)$. If there exists a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring π, such that $\overline{\pi}(u) \cap \overline{\pi}(T) \neq \emptyset$, then for any $\alpha \in \overline{\varphi}(v)$ there exists a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring σ, such that v is a $(T, \sigma, \{\alpha, \varphi(f)\})$-exit.

exit path

exit path of T:

a path intersects T at exactly one vertex.

Lemma

Suppose T is closed with respect to φ, and $f \in E(u, v)$ is an edge in $\partial(T)$ with $v \in V(T)$. If there exists a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring π, such that $\overline{\pi}(u) \cap \overline{\pi}(T) \neq \emptyset$, then for any $\alpha \in \overline{\varphi}(v)$ there exists a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring σ, such that v is a $(T, \sigma, \{\alpha, \varphi(f)\})$-exit.

a stronger result

the above holds for every $(T, C \cup \{\varphi(f)\}\}, \varphi)$-stable coloring.

defectiveness

Let T be a tree-sequence with respect to a k-triple (G, e, φ).

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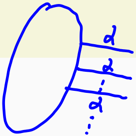
defective color α, edge $f \in \partial(T)$, and vertex v:

$|\partial_\alpha(T)| \geq 2$, $\varphi(f) = \alpha$, and v is an endvertex of f.



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$|\partial_\alpha(T)| \geq 2$, $\varphi(f) = \alpha$, and v is an endvertex of f.

free color α:

$\alpha \in \overline{\varphi}(T)$ and $\alpha \notin \varphi(T)$.

T_1

Let G be a graph such that $\chi'(G) = k + 1 \geq \Delta(G) + 2$ and e_1 be a critical edge.

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a Tashkinov tree w.r.t. e and a coloring φ satisfying the following conditions.

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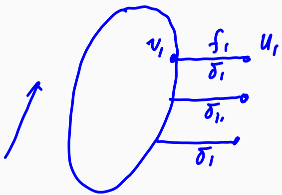
a Tashkinov tree w.r.t. e and a coloring φ satisfying the following conditions.

- $|T_1|$ is maximum among all Tashkinov trees over all k-edge colorings in $\mathcal{C}^k(G - e)$.
- T_1 is normal and closed. Assume T_1 is not strongly closed, otherwise we are done.

picking up a special edge

Along to the order $\prec$ of T, let v_1 be the largest defective vertex. Assume v_1 is maximum over all $(T_1, \emptyset, \varphi)$-stable colorings. Let $f_1 \in \partial(T_1)$ incident to v_1 such that $\pi_0(f_1) = \delta_1$ is a defective color. Let u_1 be the other endvertex of f_1.



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Case 1: $V(T) \cup \{u_1\}$ is normal for all $(T_1, \{\delta_1\}, \pi_0)$-stable colorings:
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name it SE:

Define $(T_2, \varphi_1, S_1, F_1, \Theta_1)$, where
$S_1 = \{\delta_1\}$ – connecting color
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$T_2, \varphi_1 \ldots PE$

Case 2: There is a color $\gamma_1 \in \overline{\pi}'(T_1) \cap \overline{\pi}'(u_1)$ for some $(T_1, \{\delta_0\}, \pi_0)$-stable coloring π':

We may assume $\gamma_1 \in \overline{\pi}_0'(v_1) \cap \overline{\pi}_0(u_1)$, and so path $P_1 = v_1 f_1 u_1$ is an $(T_1, \pi', \{\gamma_1, \delta_1\})$-exit path.

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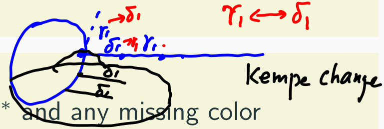

exiting property:

for any $(T_1, \{\delta_1\}, \pi_1)$-stable coloring π^* and any missing color $\gamma_1 \in \overline{\pi}^*(v_1)$, vertex v_1 is a $(T_1, \pi^*, \{\gamma_1, \delta_1\})$-exit, i.e., the (γ_1, δ_1)-chain $P_{v_1}(\gamma_1, \delta_1)$ at v_1 is a path and $V(P_{v_1}(\gamma_1, \delta_1) \cap V(T_1) = 1$.

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T_2:

pick up a $(T_1, \{\delta_1\}, \pi_0)$-stable coloring π'_0, let $\varphi_1 = \pi'_0 / P_{v_1}(\gamma_1, \delta_1)$, and let T_2 be the closure of $T_1(v_1)$ under coloring φ_1.

$T_2, \varphi_1 \ldots PE$

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maximality:

We pick up coloring π'_0 such that $|T_2|$ is maximum over all $(T_1, \{\delta_1\}, \pi_0)$-stable colorings.

T_2-II

name it PE**:**

Define $(T_2, \varphi_1, S_1, F_1, \Theta_1)$, where
$S_1 = \{\gamma_1, \delta\}$-connecting colors
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- interchangeability property: all, except one, (γ_1, δ_1)-chains intersecting T_1 are cycles;

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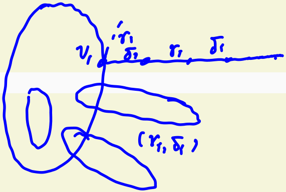


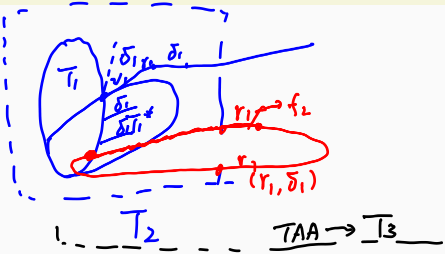
(γ_1, δ_1)**-chains:**

recall two properties:

- interchangeability property: all, except one, (γ_1, δ_1)-chains intersecting T_1 are cycles;
- exiting property: $P_{v_1}(\gamma_1, \delta_1)$ is an exiting path.
- Let $\mathcal{Q}_1$ denote the set of all (γ_1, δ_1)-cycles intersecting T_1. Then, all vertices $v \in V(T_1)\backslash\{v_1\}$ are on Q for some $Q \in \mathcal{Q}_1$.

Case R: $V(\mathcal{Q}) - V(T_2) \neq \emptyset$**:**

Let $f_2 \in \partial_{\gamma_1}(T_2)$ such that there is a path Q with colors γ_1, δ_1 in T_2 connecting f_2 and T_1. Let T_3 be the closure of $T_2 + f_2$ under the coloring φ_2 and define $S_2 = S_1$, $F_2 = \{f_1, f_2\}$ and $\Theta_3 = RE$, subject to



RE:

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- Otherwise, let φ_2^* be a (T_2, S_2, φ_1)-stable coloring with closure T_3^* such that $|T_3^*| > |T_3|$.

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- Since φ_2^* is also (T_1, S_1, φ_1)-stable, so in the extension $T_1 \to T_2$ (PE) we would have picked φ_2^* instead of φ_1, a contradiction.

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formal statements:

With the above preparation, we will give the definition of Tashkinov series $(T_n, \varphi_{n-1}, S_{n-1}, F_{n-1}, \Theta_{n-1})$, where the maximality property will be defined separately.

ETT-1

$(T_1, \varphi_0, S_0, F_0, \Theta_0)$:

$\varphi_0 = \varphi \in \mathcal{C}^k(G - e_1)$, T_1 is a closure of e by applying TAA under φ_0, and $S_0 = F_0 = \Theta_0 = \emptyset$. Suppose that we have defined $(T_n, \varphi_{n-1}, S_{n-1}, F_{n-1}\Theta_{n-1})$.

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Iteration n:

If T_n is strongly closed with respect to φ_{n-1}, stop. Else, we construct the tuple $(T_{n+1}, \varphi_n, S_n, F_n, \Theta_n)$ as follows. Set $D_{n-1} = \cup_{i \le n-1} S_i - \overline{\varphi}_{n-1}(T_{n-1})$ (so $D_0 = \emptyset$).

the condition for RE:

If there is a subscript $h \leq n-1$ with $\Theta_h = PE$ and $S_h = \{\delta_h, \gamma_h\}$, such that some (γ_h, δ_h)-cycle O with respect to φ_{n-1} intersects both $V(T_h)$ and $V(G) - V(T_n)$, apply **RE**.

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If there is a subscript $h \le n-1$ with $\Theta_h = PE$ and $S_h = \{\delta_h, \gamma_h\}$, such that some (γ_h, δ_h)-cycle O with respect to φ_{n-1} intersects both $V(T_h)$ and $V(G) - V(T_n)$, apply **RE**.

RE:

Let f_n be an edge in $O \cap \partial(T_n)$ such that O contains a path L connecting f_n and $V(T_h)$ with $V(L) \subseteq V(T_n)$. Let $\varphi_n = \varphi_{n-1}$ and T_{n+1} be a closure of $T_n + f_n$ under φ_n. Set $\delta_n = \delta_h$, $\gamma_n = \gamma_h$, $S_n = \{\delta_n, \gamma_n\}$, $F_n = \{f_n\}$, and $\Theta_n = RE$.

the condition for RE:

If there is a subscript $h \leq n-1$ with $\Theta_h = PE$ and $S_h = \{\delta_h, \gamma_h\}$, such that some (γ_h, δ_h)-cycle O with respect to φ_{n-1} intersects both $V(T_h)$ and $V(G) - V(T_n)$, apply **RE**.

Else,

let v_n be the maximum defective vertex in the order $\prec$ over all $(T_n, D_{n-1}, \varphi_{n-1})$-stable colorings, let π_{n-1} be a corresponding coloring, let f_n be a defective edge (of T_n with respect to π_{n-1}) incident to v_n, let u_n be the other end of f_n, and let $\delta_n = \pi_{n-1}(f_n)$.

SE

the condition for SE:

for every $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring π, we have $\overline{\pi}(u_n) \cap \overline{\pi}(T_n) = \emptyset$

SE

the condition for SE:

for every $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring π, we have $\overline{\pi}(u_n) \cap \overline{\pi}(T_n) = \emptyset$

SE:

Let $\varphi_n = \pi_{n-1}$ and let T_{n+1} be a closure of $T_n + f_n$ under φ_n. Set $S_n = \{\delta_n\}$, $F_n = \{f_n\}$, and $\Theta_n = SE$.

SE

the condition for SE:

for every $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring π, we have $\overline{\pi}(u_n) \cap \overline{\pi}(T_n) = \emptyset$

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Let $\varphi_n = \pi_{n-1}$ and let T_{n+1} be a closure of $T_n + f_n$ under φ_n. Set $S_n = \{\delta_n\}$, $F_n = \{f_n\}$, and $\Theta_n = SE$.

else

,

PE

the preparation for PE:

pick a color γ_n in $\overline{\pi}_{n-1}(v_n)$ as follows. If $v_n = v_i$ for some $1 \leq i < n$ with $\Theta_i = PE$, let n' be the largest such i and let $\gamma_n = \delta_{n'}$. Otherwise, let γ_n be an arbitrary color in $\overline{\pi}_{n-1}(v_n)$. Let π'_{n-1} be an arbitrary $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring so that v_n is a $(T_n, \pi'_{n-1}, \{\gamma_n, \delta_n\})$-exit

PE

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PE:

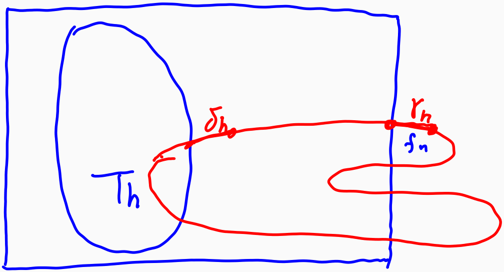
Let $\varphi_n = \pi'_{n-1}/P_{v_n}(\gamma_n, \delta_n, \pi'_{n-1})$. Let T_{n+1} be a closure of T_n under φ_n. Set $S_n = \{\delta_n, \gamma_n\}$, $F_n = \{f_n\}$, and $\Theta_n = PE$.

Extended Tashkinov tree (ETT)

ETT T of layer n:

$T_n \subseteq T \subset T_{n+1}$ for a Tashkinov Series $\{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$.

RE is good

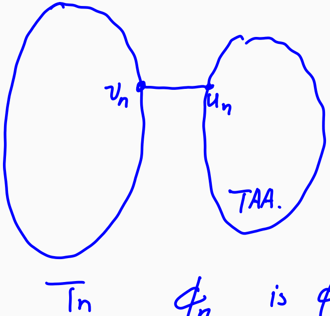


T_n

$T_n + f_n$

$\phi_{n+1} = \phi_n$ is a stable coloring

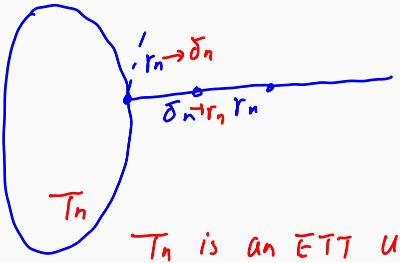
SE is good



φ_n is φ_{n-1} stable.

$\rightarrow$ T_n is an ETT under coloring φ_{n+1}

Problems with PE



T_n is an ETT under ϕ_{n-1}

but may not an ETT under ϕ_n

strongly related colorings

Let $\{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ be a Tashkinov series and $\sigma_n \in \mathcal{C}^k(G - e_1)$.

strongly related colorings

Let $\{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$ be a Tashkinov series and $\sigma_n \in \mathcal{C}^k(G - e_1)$.

σ_n: φ_n mod T_n:

if every tree-sequence $T^* \supset T_n$ obtained from $T_n + f_n$ (resp. T_n) by TAA under σ_n when $\Theta_n = RE$ or SE (resp. when $\Theta_n = PE$) is an ETT under σ_n, with a corresponding Tashkinov series $\{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$, satisfying the following conditions for all i with $1 \leq i \leq n$:

strongly related colorings

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- $T_i^* = T_i$

strongly related colorings

Let $\{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ be a Tashkinov series and $\sigma_n \in \mathcal{C}^k(G - e_1)$.

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- $T_i^* = T_i$
- σ_i is a (T_i, D_i, φ_i)-stable coloring in $\mathcal{C}^k(G - e_1)$.

strongly related colorings

Let $\{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ be a Tashkinov series and $\sigma_n \in \mathcal{C}^k(G - e_1)$.

σ_n: φ_n mod T_n:

if every tree-sequence $T^* \supset T_n$ obtained from $T_n + f_n$ (resp. T_n) by TAA under σ_n when $\Theta_n = RE$ or SE (resp. when $\Theta_n = PE$) is an ETT under σ_n, with a corresponding Tashkinov series $\{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, satisfying the following conditions for all i with $1 \le i \le n$:

- $T_i^* = T_i$
- σ_i is a (T_i, D_i, φ_i)-stable coloring in $\mathcal{C}^k(G - e_1)$.

(σ_n, T_n)-ETT:

T^* is called an ETT *corresponding* to (σ_n, T_n)

maximality property

an ETT T has the maximum property (MP) under φ_n:

if

maximality property

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if

- $|T_1|$ is maximum among all Tashkinov trees T_1' with respect to an edge $e' \in E$ and a coloring $\varphi_0' \in \mathcal{C}^k(G - e')$,

maximality property

an ETT T has the maximum property (MP) under φ_n:

if

- $|T_1|$ is maximum among all Tashkinov trees T_1' with respect to an edge $e' \in E$ and a coloring $\varphi_0' \in \mathcal{C}^k(G - e')$,
- $|T_{i+1}|$ is maximum over all (T_i, D_i, φ_i)-stable colorings for any i with $1 \leq i \leq n - 1$;

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- that is, $|T_{i+1}|$ is maximum over all tree-sequences T_{i+1}', which is a closure of $T_i + f_i$ (resp. T_i) under a (T_i, D_i, φ_i)-stable coloring φ_i' if $\Theta_i = RE$ or SE (resp. if $\Theta_i = PE$), where f_i is the connecting edge in F_i.

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Note:

In the above definition $|T_{n+1}|$ is not required to be maximum over all (T_n, D_n, φ_n)-stable colorings.

the main theorem -1

Let T be an ETT based on the Tashkinov series $\{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$. If T has MP under φ_n, then the following statements hold:

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normality:

$V(T)$ is normal with respect to φ_n.

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interchaneability

For any two colors α, β with $\overline{\varphi}_n(T_n) \cap \{\alpha, \beta\} \neq \emptyset$, there is exactly one (α, β)-path intersecting $V(T_n)$.

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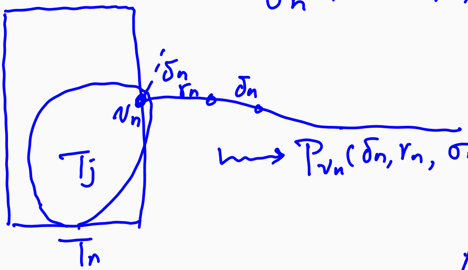
exiting path

If $\Theta_n = PE$, then $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ contains precisely one vertex, v_n, from T_n for any (T_n, D_n, φ_n)-stable coloring σ_n.

Proof of exit path property:

σ_n is $(T_n, \ldots, \phi_n)$-stable



$\rightsquigarrow P_{v_n}(\delta_n, r_n, \sigma_n) \cap T_n = \{v_n\}$

Assume $v_n \in T_j \setminus T_{j-1}$

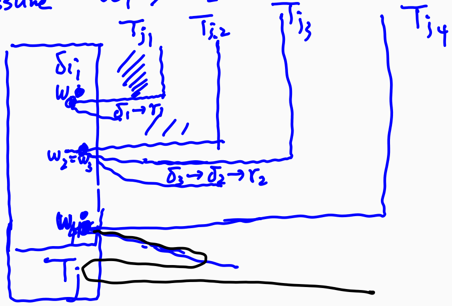
Assume. $\theta_{j_1} = \theta_{j_2} = \theta_{j_3} = \theta_{j_4} = PE.$

$j_1 < j_2 < j_3 < j_4$

With supporting vertex $w_1,\ w_2,\ w_3,\ w_4 \in T_j$

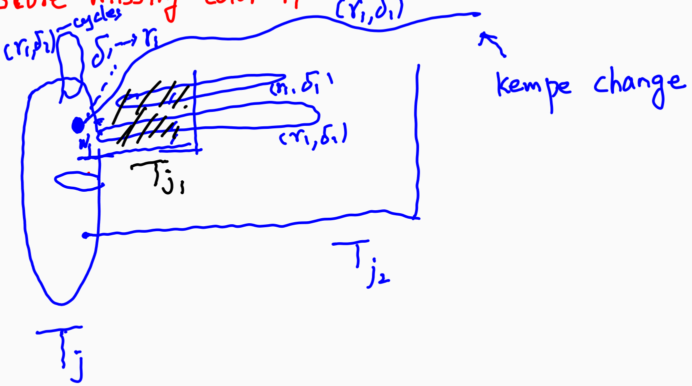
$\longrightarrow \quad w_1 \geq w_2 \geq w_3 \geq w_4.$

Assume $w_1 > w_2 = w_3 > w_4$

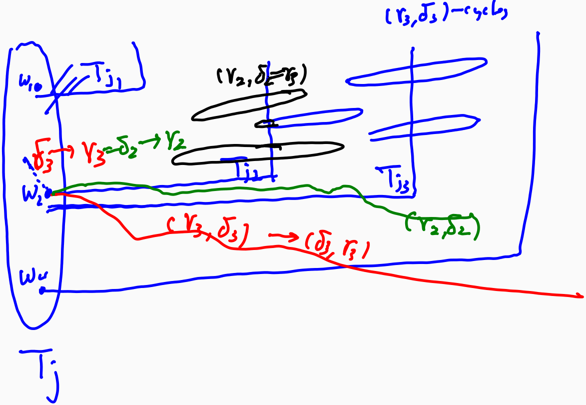


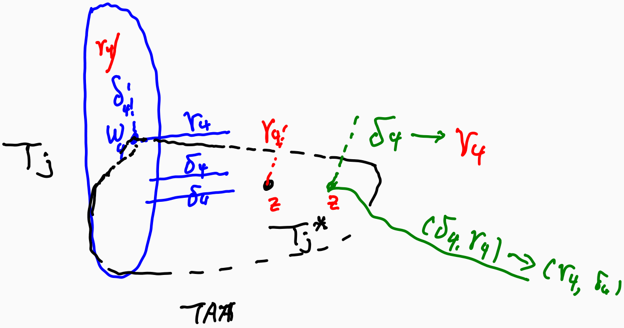
$\{r_1, \delta_1\}$, $\{r_2, \delta_1 = r_3, \delta_3\}$, $\{r_4, \delta_4\}$ disjoint.

restore missing color γ_1

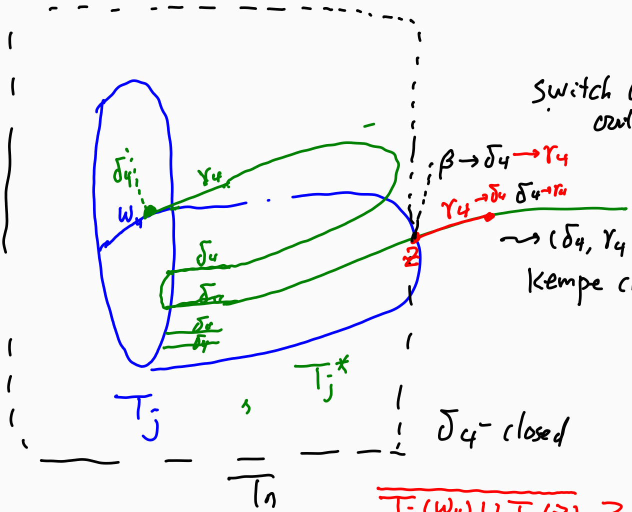


restore missing color r_2





$T_j(w_4) \cup T_j^*(z) \supsetneq T_j$ a contradiction



switch β and δ4 outside Tn

→ (δ4, γ4)-path Kempe chang

δ4-closed

$\overline{T_j(w_4) \cup T_j(z)} \not\ni T_j$

a contradiction


Thank You !