Proof of the Goldberg-Seymour Conjecture – III: a preliminary report

11th Cargèse Workshop on Combinatorial Optimization

Guantao Chen

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Georgia State University, Atlanta, US
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Overview

a few post proof thoughts

improper colorings

related results

normal, closed, and Tashkinov tree

a few post proof thoughts

behind the theorem

the theorem:

If G is a k-critical graph with $k \geq \Delta(G) + 1$, then for any edge $e \in E(G)$, graph $G - e$ is a disjoint union of k near-perfect matchings.

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- Is there a polynomial-time coloring algorithm?

improper colorings

basic definitions

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a (multi)graph G and a vertex-function $f : V(G) \to \mathbb{N}\backslash\{0\}$

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i.e., an improper (edge-)coloring φ such that $d_{\varphi,\alpha}(v) \leq f(v)$ for each vertex $v \in V(G)$ and each color α.

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$f(v) \equiv 1$:

a traditional (proper) edge-coloring of G.

applications

The f-coloring has broader applications than the ordinary edge coloring:

for example, the file transfer problem in a computer network. In the model a vertex of a graph G represents a computer, and an edge does a file which one wish to transfer between the two computers corresponding to its ends. The integer $f(v)$ is the number of communication ports available at a computer v. The edges colored with the same color represent files that can be transferred in the network simultaneously. Thus an f-coloring of G using minimum number of colors corresponds to the scheduling of the transfers with the minimum fishing time.

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one more example:

How wavelet assignment problems in so-called multi-fiber WDM networks can be modeled by means of the f color problem is explained in Koster.

more notation

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a lower bound:

$\chi'_f(G) \geq \Delta_f(G)$.

some upper bounds

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Hakimi and Kariv, 1986:

If $f(v)$ is even for every vertex $v \in V(G)$, then $\chi'_f(G) = \Delta_f(G)$.

f-density

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fractional f-density $\omega_f^*(G)$:

$$\omega_f^*(G) = \max_{H \subseteq G, |H| \geq 2} \frac{|E(H)|}{\lfloor f(H)/2 \rfloor}$$

if $|G| \geq 2$ and $\omega_f^*(G) = 0$ otherwise.

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another lower bound

$\chi_f'(G) \geq \omega_f(G)$, and so
$\chi_f'(G) \geq \max\{\Delta_f(G), \omega(G)\}$.

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the Goldberg-Seymour conjecture for f-coloring

Nakano, Nishizeki, and Saito, 1988:

$\chi'_f(G) \leq \max\{\frac{9}{8}\Delta_f(G) + \frac{6}{8}, \omega_f(G)\}$

the main result

C., and Hao, 2022+:

if $\chi'_f(G) \geq \Delta_f(G) + 2$, then $\chi'_f(G) = \omega_f(G)$.

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computing complexity?

$\omega_f(G)$ or $\max\{\Delta_f(G), \omega_f(G)\}$?

computing f-density

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a combinatorial polynomial-time algorithm for finding the fraction density $\omega^*(G)$.

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C. and Yu, 2022+:

if f is a rational value function, then there is combinatorial polynomial-time algorithm for finding the fraction f-density $\omega_f^*(G)$.

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a fractional f-coloring of G:

a function $\omega : \mathcal{M}_f(G) \to [0, 1]$ such that every edge $e \in E(G)$ satisfies $\sum_{M \in \mathcal{M}_f(G): e \in M} \omega(M) = 1$.

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the fractional f-chromatic $\chi'^{*}_{f}(G)$:

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χ'_f and χ'^*_f

$\chi'^*_f(G) \leq \chi'_f(G) \leq \chi'^*_f(G) + 1$.

related results

vertex-splitting approach

Zhou and Nishizeki, 1999:

upper bounds for the f-chromatic index of a vertex-weighted graph (G, f) can be obtained by using the following splitting operation.

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$\chi'_f(G) \leq \chi'(G_f)$:

$\Delta(G_f) = \Delta_f(G)$.

If G is simple, then $\chi'_f(G) \leq \chi'(G_f) + 1 \leq \Delta_f(G) + 1$.

not working for density

an example G:

Let G be a multi-triangle on vertices x, y, z such that
$\mu_G(x, y) = \mu_G(x, z) = t$ and $\mu_G(y, z) = 3t$ for some positive integer t.
$f(x) = 1$ and $f(y) = f(z) = 2$.
$\Delta_f(G) = d_G(x) = d_G(y)/2 = d_G(z)/2 = 2t$, and
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G_f:

Let $G_f = tK_3 \cup 2tK_2$ be a disjoint union of a multi-triangle with vertices x, y_1, z_1 and a set of $2t$ parallel edges between vertices y_2 and z_2.

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the difference is big:

$\max\{\Delta(G_f), \omega(G_f)\} - \max\{\Delta_f(G), \omega_f(G)\} = 3t - \lceil 2.5t \rceil = \lfloor 0.5t \rfloor$.

saturated and unsaturated vertices

Let φ be a partial f-coloring of a graph G, and let v be a vertex.

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$\varphi(v)$ and $\overline{\varphi}(v)$

$\varphi(v)$ the set of saturated colors at v

$\overline{\varphi}(v)$ the set of unsaturated colors at v

"Kempe-chains" in a improper edge-coloring

Let φ be a proper edge-coloring of G, and α, β be two distinct colors.

"Kempe-chains" in a improper edge-coloring

Let φ be a proper edge-coloring of G, and α, β be two distinct colors.

$G_{\alpha,\beta} = G[E_{\alpha,\beta}]$:

a disjoint union of paths and even cycle.

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φ is not a proper edge-coloring?

Goal:decompose $G_{\alpha,\beta}$ to some desired subgraphs.

alternating trails and alternating walks

Let φ be an f-coloring of G and α, β be two distinct colors.

$\overline{d}_\alpha(v)$:

$\overline{d}_\alpha(v) = f(v) - d_\alpha(v)$

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a trail whose edges are alternately colored with colors α, β.

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a nontrivial $\alpha\beta$-alternating walk $W = (v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$:

(1) $\varphi(e_1) = \beta$;

(2) $\overline{d}_\alpha(v_0) \geq 1$ if $v_0 \neq v_p$; $\overline{d}_\alpha(v_0) \geq 2$ if W is an odd circuit; and $\overline{d}_\alpha(v_0) \geq 1 \leq \overline{d}_\beta(v_0)$ if W is an even circuit;

(3) $\overline{d}_\beta(v_p) \geq 1$ if $v_0 \neq v_p$ and $\varphi(e_p) = \alpha$ and $\overline{d}_\alpha(v_p) \geq 1$ if $v_0 \neq v_p$ and p is odd.

a nontrivial (α, β)-walk:

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end(vertices):

trivial (α, β)-walk – even circuit – does not have ends (endvertices)

(α, β)-system

an (α, β)-alternating trail system ((α, β)-ATS) $\mathcal{W}$:

a family of edge-disjoint (α, β)-trails

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- all trails in $\mathcal{W}$ are (α, β)-walks,

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a family of edge-disjoint (α, β)-trails

an (α, β)-alternating walk system ((α, β)-AW) $\mathcal{W}$:

- all trails in $\mathcal{W}$ are (α, β)-walks,
- at each vertex $v \in V(G)$, the family $\mathcal{W}$ contains at most $\overline{d}_\beta(v)$ edges colored α that end at v

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an (α, β)-alternating walk system ((α, β)-AW) $\mathcal{W}$:

- all trails in $\mathcal{W}$ are (α, β)-walks,
- at each vertex $v \in V(G)$, the family $\mathcal{W}$ contains at most $\overline{d}_\beta(v)$ edges colored α that end at v
- at each vertex $v \in V(G)$, the family $\mathcal{W}$ contains $\overline{d}_\alpha(v)$ edges colored β that ends at v.

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covering system

Let $F \subseteq E(G)$, and let $\mathcal{F}, \mathcal{W}$ be families of edge-disjoint subgraphs of G.

$\mathcal{W}$ covers F:

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Lemma

Let φ be an f-coloring of a weighted graph (G, f). For every (α, β)-ATS $\mathcal{F}$, there is an (α, β)-ARS that covers $\mathcal{F}$.

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Corollary

For any edge set $F \subseteq E_{\alpha,\beta}(G)$, there exists an (α, β)-ARS covering F. Consequently, for every α- or β-edge there is an (α, β)-walk containing it.

covering with preserved properties

Definition

Let $\mathcal{F}$ be an (α, β)-ATS and $E^d \subseteq E(\mathcal{F})$ be a set of end-edges of $\mathcal{F}$. A vertex v is $(\mathcal{F}, E^d)$-attainable if $\mathcal{F}$ contains at most $\overline{d}_\beta(v)$ edges colored α in E^d that end at v and at most $\overline{d}_\alpha(v)$ edges colored β in E^d that end at v. An (α, β)-ATS $\mathcal{W}$ strongly covers $(\mathcal{F}, D^d)$ if it covers $\mathcal{F}$ and at any $(\mathcal{F}, E^d)$-attainable vertex v, every edge in E^d remains an end-edge in $\mathcal{W}$.

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Let $\mathcal{F}$ be an (α, β)-ATS and $E^d \subseteq E(\mathcal{F})$ be a set of end-edges of $\mathcal{F}$. A vertex v is $(\mathcal{F}, E^d)$-attainable if $\mathcal{F}$ contains at most $\overline{d}_\beta(v)$ edges colored α in E^d that end at v and at most $\overline{d}_\alpha(v)$ edges colored β in E^d that end at v. An (α, β)-ATS $\mathcal{W}$ strongly covers $(\mathcal{F}, D^d)$ if it covers $\mathcal{F}$ and at any $(\mathcal{F}, E^d)$-attainable vertex v, every edge in E^d remains an end-edge in $\mathcal{W}$.

Lemma

For any (α, β)-ATS $\mathcal{F}$ and end-edge set $E^d \subseteq E(\mathcal{F})$, there is an (α, β)-AWS $\mathcal{W}$ that strongly covers $(\mathcal{F}, E^d)$.

normal, closed, and Tashkinov tree

Normal, closed, and strongly closed

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U strongly closed:

Colors appearing on the boundary are distinct.

goal

Let (G, f) be a weighted graph and $e \in E(G)$ such that $(\chi'_f(G) = k + 1$ and $\chi'_f(G - e) = k$, and $k \geq \Delta_f(G) + 1$.

goal

Let (G, f) be a weighted graph and $e \in E(G)$ such that $(\chi'_f(G) = k + 1$ and $\chi'_f(G - e) = k$, and $k \geq \Delta_f(G) + 1$.

Goal:

Find a coloring $\varphi \in \mathcal{C}^k_f(G - e)$ and a vertex set $U \subseteq V(G)$ such that $e \in E[U]$ and U is both normal and strongly closed.

Tashkinov tree and extended Tashkinov tree

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all Tashkinov trees are normal.

Tashkinov tree and extended Tashkinov tree

Tashkinov tree:

all Tashkinov trees are normal.

extended Tashkinov tree:

There is an extension of Tashkinov tree which is both normal and strongly closed.


Thank You !