Cargese Comb Opt I:
Stable & log-concave generating polynomials

Notation: work in $\mathbb{R}[x_1,\dots,x_n]$, $\partial_i = \frac{\partial}{\partial x_i}$

For $\alpha \in \mathbb{Z}^n_{\geq 0}$, $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $\partial^\alpha := \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$

For $\alpha = \mathbb{1}_S \in \{0,1\}^n$, use $x^S = x^{\mathbb{1}_S}$, $\partial^S = \partial^{\mathbb{1}_S}$

$\mu : \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{\geq 0} \to$ generating function $g_\mu = \sum_\alpha \mu(\alpha) x^\alpha$

$\text{supp}(\mu) = \text{supp}(g_\mu) = \{\alpha : \mu(\alpha) \neq 0\}$ finite

Note: $\text{Prob}_\mu(\alpha) = \dfrac{\mu(\alpha)}{g_\mu(\mathbb{1})}$ is a discrete prob. dist.

Often g_μ is homog. $(\alpha_1 + \dots + \alpha_n = d$ for all $\alpha \in A)$

Important special case: $\text{supp}(\mu) \subseteq \{0,1\}^n$

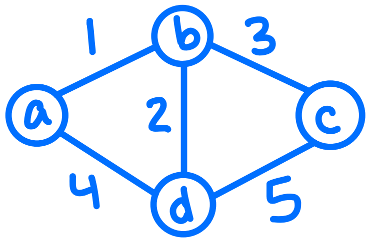
$g_\mu = \sum_{S \subseteq [n]} \mu(S) x^S$

$\partial_i g_\mu = \sum_{S \ni i} \mu(S) x^{S \setminus i}$ (conditioning on $i \in S$)

$g_\mu|_{x_i=0} = \sum_{S \not\ni i} \mu(S) x^S$ (conditioning on $i \notin S$)

Ex 0: $g = x_1x_2 + x_1x_3 + x_2x_3$ $\quad \partial_1 g = x_2 + x_3$

Ex 1: $g = \sum_T \prod_{e \in T} x_e$ where T runs over all spanning trees of a graph



$\rightarrow \; x_1x_2x_3 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_2x_3x_4 + x_2x_4x_5 + x_3x_4x_5$

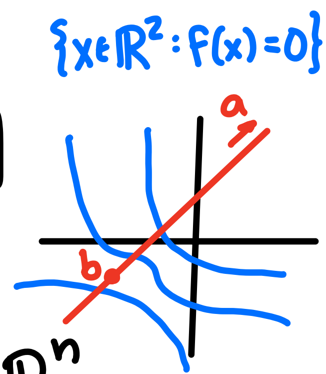
Class of polynomials $g_\mu \longleftrightarrow$ class of distributions μ

Want

- closure under natural operations
- include important examples
- implications for "shape" of μ
- efficient algorithms for (approx) sampling

{Real stable poly} $\subseteq$ { (strongly) log-concave poly. }

<u>Real stability</u>



$f \in \mathbb{R}[x_1, \dots, x_n]$ is <u>stable</u> if $f(ta+b) \in \mathbb{R}[t]$ is real rooted for all $a \in \mathbb{R}^n_+$, $b \in \mathbb{R}^n$

Equiv: $f(z) \neq 0$ for all $z \in \mathbb{C}^n$ with $\operatorname{Im}(z) \in \mathbb{R}^n_+$

<u>Ex</u>: $f = \prod_{i=1}^n x_i \qquad f(ta+b) = \prod_{i=1}^n (ta_i + b_i)$

<u>Ex</u>: $D_a f = \sum_{i=1}^n a_i \partial_i f$ where f stable, $a \in \mathbb{R}^n_{\geq 0}$

e.g. $f = \prod_{i=1}^n x_i$, $a = (1, \dots, 1)$, $D_a f = \sum_{i=1}^n \prod_{j \neq i} x_j = e_{n-1}(x_1, \dots, x_n)$

Ex: $f = \det(x_1 A_1 + \dots + x_n A_n)$ where $A_1, \dots, A_n \in \mathbb{R}^{d\times d}_{sym}$ are PSD

Aside: $M \in \mathbb{R}^{d\times d}_{sym}$ is positive semidefinite (PSD)
$\iff$ all eigenvalues of M are ≥ 0
$\iff$ $M = UU^T$ for some $U \in \mathbb{R}^{d\times m}$

Why? $f(ta+b) = \det(t\underline{A(a)} + A(b))$ $\quad A(x) = \sum_i x_i A_i$

pos. def $\Rightarrow$ $A(a) = UU^T$, $U \in \mathbb{R}^{d\times d}$ invertible

$$= \det(U)^2 \det(tI + U^{-1}A(b)U^{-T})$$

roots $= -($eigval of $U^{-1}A(b)U^{-T} \in \mathbb{R}^{d\times d}_{sym})$

e.g. $f = \det\begin{pmatrix} x_1+x_3 & x_3 \\ x_3 & x_2+x_3 \end{pmatrix} = x_1x_2 + x_1x_3 + x_2x_3$

e.g. $f = \det\left(\sum_{i=1}^n x_i v_i v_i^T\right)$ where $v_1, \dots, v_n \in \mathbb{R}^d$

$$= \sum_{S \in \binom{[n]}{d}} \det(v_i : i \in S)^2 \, x^S$$

e.g. $f = \sum_T \prod_{e\in T} x_e$ where T runs over all spanning trees of a graph

(weighted matrix tree theorem)

Borcea and Brändén classify linear operators on $\mathbb{R}[x_1, \dots, x_n]$ preserving stability.

Connections to negative correlation

<u>Thm</u> (Brändén, 2007) If $f \in \mathbb{R}[x_1, \dots, x_n]$ is stable, then for every $i, j \in [n]$,

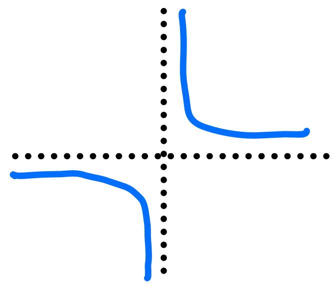
$$\Delta_{ij}(f) = \partial_i f \cdot \partial_j f - f \cdot \partial_i \partial_j f$$

is nonnegative on $\mathbb{R}^n$.

Idea: $f = a x_1 x_2 + b x_1 + c x_2 + d$ stable

$\Leftrightarrow \Delta_{12} f = bc - ad \geq 0$

$= (a x_2 + b)(a x_1 + c) - f \cdot a$



<u>Ex</u>: $f = x_1 x_2 + x_1 x_3 + x_2 x_3$

$\Delta_{12}(f) = (x_2 + x_3)(x_1 + x_3) - f \cdot 1 = (x_3)^2$

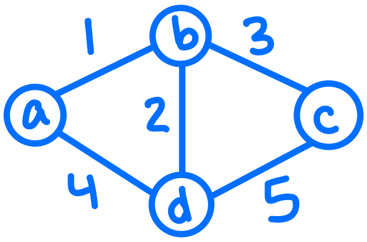
generating polynomials:

$g_\mu = \sum_{S \subseteq [n]} \mu(S) x^S$ stable, $g_\mu(\mathbb{1}) = 1$, $\text{Prob}_\mu(S) = \mu(S)$

$\partial_i g_\mu = \sum_{S \ni i} \mu(S) x^{S \setminus i} \Rightarrow \partial_i g_\mu(\mathbb{1}) = \text{Prob}(i \in S)$

$$\Delta_{ij}(g_\mu)(\mathbb{1}) = \underbrace{\text{Prob}(i \in S)\,\text{Prob}(j \in S) - \text{Prob}(i, j \in S) \geq 0}_{\text{"negative correlation"}}$$

Ex: Pick a spanning tree of a graph uniformly at random



$$f = \frac{1}{8}\left(x_1x_2x_3 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_2x_3x_4 + x_2x_4x_5 + x_3x_4x_5\right)$$

$\mathrm{Prob}(1 \in T) = 5/8$, $\mathrm{Prob}(2 \in T) = 4/8$, $\mathrm{Prob}(1,2 \in T) = \frac{2}{8} < \frac{4}{8}\cdot\frac{5}{8}$

"g_μ stable" $\Leftrightarrow$ μ is "strongly Rayleigh"

"Negative dependence and the geometry of polynomials" by Borcea, Brändén, Liggett

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Connections to matroids

Thm (Choe, Oxley, Sokal, Wagner, 2004)

If $f = \sum_{S \in \binom{[n]}{d}} c_S x^S$ is stable then $\mathcal{B} = \{S : c_S \neq 0\}$ are the bases of a matroid.

Brändén: Not all matroids come this way! (Fano matroid)

Matroids

$\mathcal{B}$ = nonempty collection of subsets of $[n]$

$M = ([n], \mathcal{B})$ is a matroid if

$A, B \in \mathcal{B},\ a \in A \setminus B \Rightarrow \exists\, b \in B \setminus A$ s.t. $(A \setminus a) \cup \{b\} \in \mathcal{B}$

"rank" of M

$\mathcal{B}$ = "bases" of M (note $|A|=|B|$ for all $A,B\in\mathcal{B}$)

$\mathcal{I}$ = "independent sets" = $\{I\subseteq[n] : \exists B\in\mathcal{B} \text{ with } I\subseteq B\}$

Log concave polynomials

Let $f\in\mathbb{R}_{\geq 0}[x_1,\ldots,x_n]$. ($f=\sum_\alpha c_\alpha x^\alpha$ with $c_\alpha\in\mathbb{R}_{\geq 0}$)

f is log-concave on $\mathbb{R}^n_+$ if $\log(f):\mathbb{R}^n_+\to\mathbb{R}$ is concave

That is, $\nabla^2\log(f) = (\partial_i\partial_j \log(f))_{ij}$ is negative semidefinite at every $x=a\in\mathbb{R}^n_+$.

all eigenvalues ≤ 0

equivalent: $v^T\nabla^2\log(f)v = D_v^2\log(f)\leq 0 \;\; \forall v\in\mathbb{R}^n$

f is strongly log concave (SLC) if for all $\alpha\in\mathbb{Z}^n_{\geq 0}$, $\partial^\alpha f$ is log-concave on $\mathbb{R}^n_+$

also called Lorentzian for homog. poly

Ex: Real rooted polynomials $p(t)=(t-r_1)\cdots(t-r_d)$

$\log(p)=\sum_{i=1}^d \log(t-r_i) \Rightarrow \log(p)'' = \sum_{i=1}^d \frac{-1}{(t-r_i)^2} \leq 0$ for $t\in\mathbb{R}$

Ex: Homogeneous stable polynomials

Ex/Thm (Anari-Liu-Oveis Gharan-V., Brändén-Huh)

$$f=\sum_{B\in\mathcal{B}} x^B \quad \text{and} \quad g_{\mathcal{I}}=\sum_{I\in\mathcal{I}} x^I y^{n-|I|}$$

for any matroid $([n], \mathcal{B})$

<u>Cor</u> (Almost negative dependence)

$$[\nabla^2 \log(f)]_{ij,ij} = \frac{1}{f^2}\begin{bmatrix} -(\partial_i f)^2 & * \\ f\,\partial_{ij} f - \partial_i f \cdot \partial_j f & -(\partial_j f)^2 \end{bmatrix}$$

$$\det(\uparrow) = \frac{\partial_i \partial_j f}{f^3}\left(2\partial_i f \cdot \partial_j f - f \cdot \partial_i \partial_j f\right) \geq 0$$

eval at $x = \mathbb{1}$ $\rightarrow$ $2\,\mathrm{Prob}(i \in B)\,\mathrm{Prob}(j \in B) \geq \mathrm{Prob}(i,j \in B$

See Huh-Schröter-Wang for more.

What is $\nabla^2 \log(f)$? $\nabla^2 \log(f) = \dfrac{f \cdot \nabla^2 f - \nabla f \nabla f^T}{f^2}$

$\nabla f^T = (\partial_1 f, \dots, \partial_n f)$, $\nabla^2 f = (\partial_i \partial_j f)_{ij}$

<u>Ex</u>: $f = x_1x_2 + x_1x_3 + x_2x_3$ $\quad \nabla f^T = (x_2 + x_3,\ x_1 + x_3,\ x_1 + x_2)$

$$\nabla^2 f = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad f^2 \cdot \nabla^2 \log(f) = \begin{pmatrix} -(x_2+x_3)^2 & -x_3^2 & -x_2^2 \\ -x_3^2 & -(x_1+x_3)^2 & -x_1^2 \\ -x_2^2 & -x_1^2 & -(x_1+x_2)^2 \end{pmatrix}$$

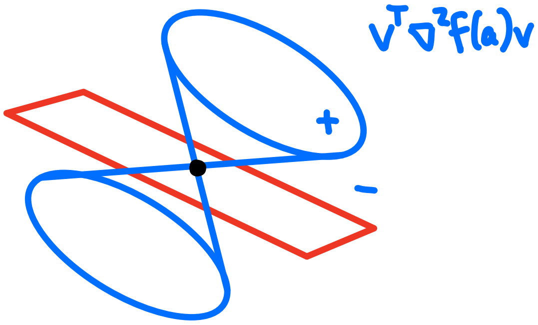
<u>Lemma</u>: For $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be homog. of deg d.

f is log concave at $x = a \in \mathbb{R}^n_+$

$\iff$ $\nabla^2 f$ at $x = a$ has ≤ 1 positive eigenvalue

(Proof $\Rightarrow$) f log concave $\Rightarrow v^T\left(f(a)\nabla^2 f(a) - \nabla f(a) \nabla f(a)^T\right) v \leq 0$

$\Rightarrow v^T \nabla^2 f(a) v \leq 0$ for v with $\nabla f(a)^T v = 0$



$\Rightarrow$ hyperplane $\{v : \nabla f(a)^T v = 0\}$ intersects span eigenvectors of $\nabla^2 f(a)$ with positive eigenvalues only in $\{0\}$. $\square$

<u>Newton's ineq</u>: $f = \sum_{k=0} c_k x_1^k x_2^{d-k}$ SLC

$$\Rightarrow \left(\frac{c_k}{\binom{d}{k}}\right)^2 \geq \left(\frac{c_{k-1}}{\binom{d}{k-1}}\right)\left(\frac{c_{k+1}}{\binom{d}{k+1}}\right)$$

discrete log-concavity of coeff.

(Proof) $q = \partial_1^{k-1} \partial_2^{d-k-1} f \in \mathbb{R}[x_1, x_2]_2$ is log-concave

$$\Rightarrow 0 \geq \det(\nabla^2 q) = \det\begin{pmatrix} \partial_1^{k+1}\partial_2^{d-k-1} f & \partial_1^k \partial_2^{d-k} f \\ \partial_1^k \partial_2^{d-k} f & \partial_1^{k-1}\partial_2^{d-k+1} f \end{pmatrix}$$

$$= (d!)^2 \det\begin{pmatrix} c_{k+1}/\binom{d}{k+1} & c_k/\binom{d}{k} \\ c_k/\binom{d}{k} & c_{k-1}/\binom{d}{k-1} \end{pmatrix}$$

<u>Cor</u> (Mason Conj) $g(t,..,t,y) = \sum_{k=0}^{n} i_k t^k y^{n-k}$ is SLC

$i_k = \#\{I \in \mathcal{I} : |I| = k\}$ $\Rightarrow \left(\frac{i_k}{\binom{n}{k}}\right)^2 \geq \left(\frac{i_{k+1}}{\binom{n}{k+1}}\right) \cdot \left(\frac{i_{k-1}}{\binom{n}{k-1}}\right)$

Cargese Comb Opt II

Log-concavity & Markov chains

Let $g = \sum_\alpha \mu(\alpha) x^\alpha \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be homog. of deg. d

(e.g. $g = \sum_{B \in \mathcal{B}} x^B$ where $([n], \mathcal{B})$ is a matroid)

<u>Def</u>: g is <u>indecomposable</u> if the graph with vertices $\{i : \partial_i f \neq 0\}$ and edges $\{\{i,j\} : \partial_i \partial_j f \neq 0\}$ is connected.

ex. $x_1x_2 + x_2x_3$ non-ex: $x_1x_2 + x_3x_4$

 ✓  ✗

Thm (ALOV, Brändén-Huh) TFAE:

① $\partial^\alpha g$ is log-concave on $\mathbb{R}^n_+$ $\forall \alpha \in \mathbb{Z}^n_{\geq 0}$ (strong log-concavity)

② $D_{v_1} \cdots D_{v_k} g$ is log-concave on $\mathbb{R}^n_+$ $\forall v_1, \dots, v_k \in \mathbb{R}^n_{\geq 0}$ (CLC)

③ (i) $\partial^\alpha g$ is indecomposable for all $|\alpha| \leq d-3$

(ii) $\partial^\alpha g$ is log-concave for all $|\alpha| = 2$

④ (i) $\mathrm{Newt}(g)$ is M-convex and $\mu(\alpha) \neq 0$ $\forall \alpha \in \mathrm{Newt}(g)$

(ii) $\nabla^2 \partial^\alpha g$ has ≤ 1 pos. eig. value $\forall |\alpha| = d-2$

③/④ easiest to check

② related to comb. Hodge theory (Adiprasito-Huh-Katz)

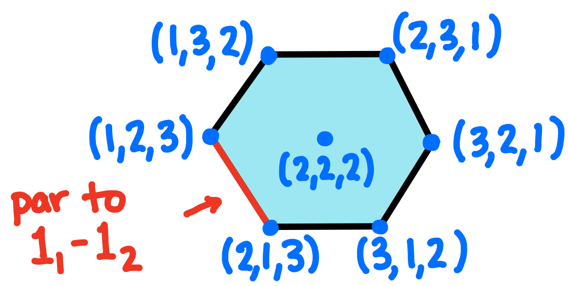
$\Rightarrow$ closure under $g \mapsto g(x_1v_1+\dots+x_kv_k)$ for $v_1,\dots,v_k \in \mathbb{R}^n_{\geq 0}$

e.g. specializing to $x_i = x_j$

① useful for Markov chains

A polytope $P \subseteq \mathbb{R}^n$ is M-convex (= generalized permutohedron) if every edge of P is parallel to $1_i - 1_j$ for some $i,j \in [n]$.

Ex: permutohedra $= \mathrm{conv}\{(\pi(1),\pi(2),\pi(3)) : \pi \in S_3\}$



$\{x : x_1+x_2+x_3 = 6\}$

Goal for today: Sketch ③ $\Rightarrow$ ②

Moral: all about quadratic forms

Lemma 1: f is CLC $\Leftrightarrow$ $\forall\, v_1,\dots,v_{d-2} \in \mathbb{R}^n_{\geq 0}$,

$D_{v_1}\cdots D_{v_{d-2}} f$ is LC on $\mathbb{R}^n_+$ $\leftarrow$ homog of deg 2

(idea: Euler's formula: $D_v f|_{x=v} = d \cdot f(v)$)

Lemma 2: Let $q(x) = \frac{1}{2}x^TQx$ with $Q \in (\mathbb{R}_{\geq 0})^{n\times n}_{\text{sym.}}$

For $a \in \mathbb{R}^n_+$ with $q(a) \neq 0$, the following are equivalent:

1) q is log concave on $\mathbb{R}_+^n$

2) $q(x) \leq 0$ for all $x \in \nabla q(a)^\perp = Qa^\perp$

3) $q(x) \leq 0$ for all x in some hyperpane H

4) Q has ≤ 1 positive eigenvalue ("Lorentzian signature")

(Proof) (3)$\Rightarrow$(1) Take $b \in \mathbb{R}^n$.

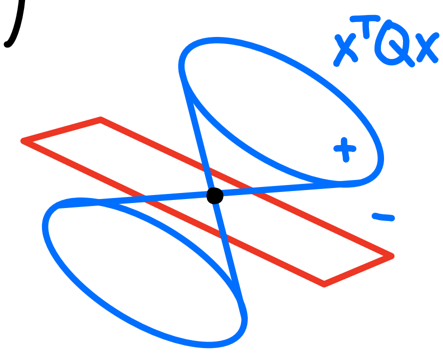
$$\nabla^2 \log(q)(a) = \frac{1}{q(a)^2}\left(q(a)Q - \nabla q(a) \nabla q(a)^T \right)$$

<u>Claim</u>: $\det\left(\begin{bmatrix} - a^T - \\ - b^T - \end{bmatrix} Q \begin{bmatrix} | & | \\ a & b \\ | & | \end{bmatrix}\right) = \det\begin{pmatrix} a^TQa & a^TQb \\ b^TQa & b^TQb \end{pmatrix} \leq 0$

- $a^TQa = 2q(a) > 0 \Rightarrow$ this 2×2 matrix not NSD
- $\mathrm{span}_{\mathbb{R}}\{a,b\} \cap H$ has dim ≥ 1 $\ni v_1 a + v_2 b$

$\Rightarrow \exists v \in \mathbb{R}^2 \setminus \{0\}$ s.t. $v^T\begin{pmatrix} a^TQa & a^TQb \\ b^TQa & b^TQb \end{pmatrix} v \leq 0$

$\Rightarrow$ this 2×2 matrix not PSD



$\lambda_1 \geq 0,\ \lambda_2 \leq 0 \Rightarrow \lambda_1 \cdot \lambda_2 = \det \leq 0$

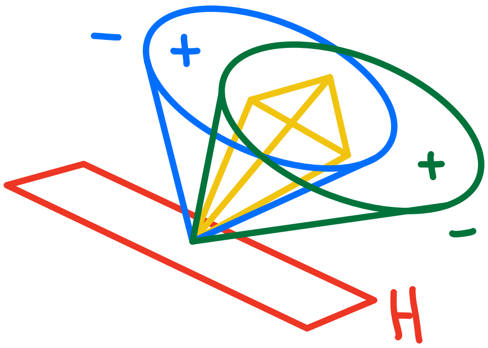
$$\Rightarrow 0 \geq (a^TQa)(b^TQb) - (b^TQa)(a^TQb)$$
$$= b^T\left(\boxed{(a^TQa)Q - (Qa)(Qa)^T}\right)b$$

NSD

$$\nabla^2 \log(q)(a) = \frac{2}{q(a)^2}\boxed{\uparrow} - (Qa)(Qa)^T \text{ also NSD}$$

<u>Lemma 3</u>: f, g CLC, $D_a f = D_b g \neq 0$ for $a, b \in \mathbb{R}_+^n$

$\Rightarrow f + g$ CLC



Idea: f, g quadratic
$f, g \le 0$ on $H = D_a f^{\perp} = D_b g^{\perp}$
$\Rightarrow f + g \le 0$ on H.

<u>Lemma 4</u>: f indecomposable, $\partial_1 f, \dots, \partial_n f$ CLC
$\Rightarrow D_v f$ CLC for any $v \in \mathbb{R}^n_+$

<u>Claim</u>: $\sum_{i=1}^{k} v_i \partial_i f$ CLC for all $k = 1, \dots, n$

Induct on k. $a = (v_1, \dots, v_k, 0, \dots, 0)$, $b = (0, \dots, 0, v_{k+1}, 0, \dots, 0)$

$D_a D_b f = D_b D_a f = \sum_{i=1}^{k} v_i v_{k+1} \partial_i \partial_{k+1} f \Rightarrow D_a f + D_b f = \sum_{i=1}^{k+1} v_i \partial_i f$ CLC

Inducting on $\deg(g)$ gives proof of ③ ⇒ ②

<u>Back to matroids</u>:

$M = ([n], \mathcal{B})$ a matroid, $g_M = \sum_{B \in \mathcal{B}} x^B$

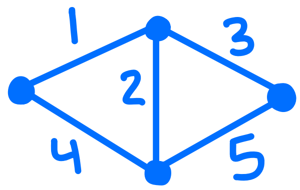
For $S \subseteq [n]$, $\partial^S g_M = \sum_{B \in \mathcal{B} : S \subseteq B} x^{B \setminus S} = g_{M/S}$ $\qquad |S| = d-2$

M/S rank 2: $i \sim j \Leftrightarrow \{i,j\} \notin \mathcal{B}$ equivalence relation
$(\{i,j\}, \{j,k\} \notin \mathcal{B} \Rightarrow \{i,k\} \notin \mathcal{B})$

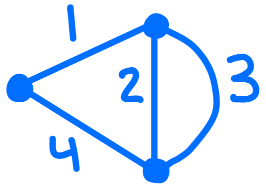
$$\Rightarrow \nabla^2 g_{M/S} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \mathbb{1}\mathbb{1}^T - \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{PSD}}$$

$\Rightarrow g_{M/S}(x) \le 0$ on $\mathbb{1}^\perp \Rightarrow$ log-concave on $\mathbb{R}^n_+$

+ indecomposability $\Rightarrow g_M$ is CLC

Ex: 

$g_M = x_1x_2x_3 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_2x_3x_4 + x_2x_4x_5 + x_3x_4x_5$



$\partial_5 g_M = g_{M/5} = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_4 + x_3x_4$

$$\nabla^2(\partial_5 g_M) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \mathbb{1}\mathbb{1}^T - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\tfrac{1}{2} x^T(\nabla^2 \partial_5 g)x = \partial_5 g = (x_1+x_2+x_3+x_4)^2 - x_1^2 - (x_2+x_3)^2 - x_4^2$$

≤ 0 when $x_1+x_2+x_3+x_4 = 0$

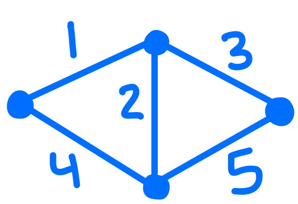
Random walk on $\mathcal{B}$ $(N = |\mathcal{B}|)$

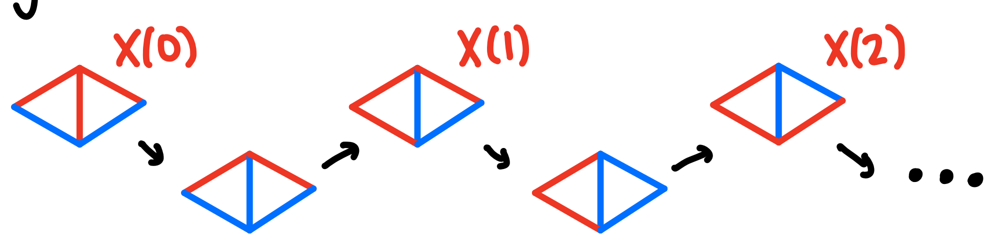
$X(t) = B \in \mathcal{B} \rightarrow$ "down-up walk"

1) Remove $b \in B$ uniformly at random
2) Add $a \in [n]$ s.t. $B \setminus \{b\} \cup \{a\} \in \mathcal{B}$ uniformly at random

$X(t+1) = B \setminus \{b\} \cup \{a\}$ "up-down walk"

Repeating (2),(1) gives walk on $\{I \in \mathcal{I} : |I| = d-1\}$

Ex: 



Ex: $B = \{12, 13, 14, 23, 24, 34\}$

$$\{i,j\} \xrightarrow{1/2} \{i\} \xrightarrow{1/3} \{i,k\} \xrightarrow{1/2} \{k\}$$

$$P^{\vee} = \begin{array}{c} \\ 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{pmatrix} 1/3 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \\ 1/6 & 1/3 & 1/6 & 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 1/3 & 0 & 1/6 & 1/6 \\ 1/6 & 1/6 & 0 & 1/3 & 1/6 & 1/6 \\ 1/6 & 0 & 1/6 & 1/6 & 1/3 & 1/6 \\ 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/3 \end{pmatrix} \end{array} = \underbrace{\tfrac{1}{6} A^T A}_{\lambda_6 \geq 0} \qquad A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Transition matrix for "up-down" walk on $\{1,2,3,4\}$ is

$$P^{\wedge} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/2 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/2 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/2 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/2 \end{pmatrix} \end{array} = \frac{1}{6} A A^T = \underbrace{\frac{1}{2} I + \frac{1}{6} \left(\boxed{\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}} \right)}_{\lambda_2 \leq 1/2}$$

Fact: $A^T A$, $A A^T$ have same nonzero eig.vals

$\Rightarrow \lambda_2(P^{\vee}) \leq 1/2$ $\qquad \nabla^2(x_1 x_2 + \ldots + x_3 x_4) = \nabla^2(f_M)$

$\qquad \Rightarrow \lambda_2 \leq 0$

$\Rightarrow \lambda^*(P) = \max\{\lambda_2, |\lambda_6|\} \leq 1/2$

Thm (Diaconis, Stroock '91) P = trans. matrix

$$\text{mix time} \leq \frac{1}{1 - \lambda^*(P)} \cdot \log\left(\frac{1}{\varepsilon \cdot v^*}\right)$$

where $\lambda^*(P) = \max\{\lambda_2, |\lambda_N|\}$, $v^* = \min_j \{v_j\}$

$\Rightarrow$ want $\lambda^*(P) \ll 1$ for fast mixing

High Dimensional Expanders

Δ = simp'l complex, pure of dim $d-1$

w = weight on maximal faces

Main e
$\Delta = I$
$W(B)$

?

For $\sigma \in \Delta$, $\mathrm{link}_\sigma(\Delta) = \{\tau \setminus \sigma : \tau \in \Delta, \sigma \subseteq \tau\}$

Thm (Kauffman-Oppenheim, '18) If

(i) all links of Δ with dim ≥ 1 have connected 1-skeleton

(ii) trans. matrix for "up-down walk" on 1-dim'l links of Δ has $\lambda_2 \leq 1/2$

then trans. matrix for "down-up walk" on $(d-1)$-dim'l faces has $\lambda_2 \leq 1 - \frac{1}{d}$

Idea: consider Markov chains on $\Delta(k) \cup \Delta(k-1)$

$$\begin{tikzcd} \Delta(k) & S \arrow[dr] & & S\setminus i \cup j \arrow[dr] & \\ \Delta(k-1) & & S\setminus i \arrow[ur] & & \cdots \end{tikzcd}$$

Trans. matrices

"down-up" on $\Delta(k)$: $P_k^\vee$

"up-down" on $\Delta(k-1)$: $P_{k-1}^\wedge$

$P_k^\vee$, $P_{k-1}^\wedge$ have same nonzero eig. val., all ≥ 0

bound eig. val of $P_k^\vee$ using eig. val of $P_k^\wedge$

Start w/ $\lambda_2(P_1^\wedge) = \lambda_2(P_2^\vee) \leq 1/2$ and induct

Thm (ALOV '19) (Δ, w) satisfies (i) and (ii)

$\iff f = \sum_{\sigma \in \Delta(d-1)} w(\sigma) x^\sigma$ is SLC

$\Delta = \mathcal{I}$ $w(I) = 1$

Cor: For any matroid of rank d, this random walk on $\mathcal{I}(d-1) = \mathcal{B}$ mixes in time $O(d^2 \log(n))$

→ later improved to $O(d \log(d))$

Sample $B \in \mathcal{B}$ by starting at any $X(0) = A \in \mathcal{B}$ taking $O(d \log(d))$ steps, output $X(t) = B \in \mathcal{B}$.

Cargese Comb Opt III

Log-concave Polynomials: Overview

- "easy" to test
- satisfied by many interesting ex
  - basis gen. poly (polymatroids/M-convex funct.) Brändén - Huh "Lorentzian poly"
  - $\mathrm{Vol}(x_1K_1+..+x_nK_n)$, $K_1,..,K_n \subseteq \mathbb{R}^d$ convex bodies
  - multivariate Alexander poly of special alt. links Hafner-Mészáros-Vidinas
- closed under many operations
  multiplication, specialization $f \to f|_{x_i=x_j}$, ...
- implies discrete log concavity of coeff

  e.g. $g=\sum_{I\in\mathcal{I}} \prod_{i\in I} x_i\, y^{n-|I|}$ check SLC using $\partial^\alpha g$

  $g|_{x_1=..=x_n=x} = \sum i_k x^k y^{n-k} \Rightarrow \left(\frac{i_k}{\binom{n}{k}}\right)^2 \geq \left(\frac{i_{k-1}}{\binom{n}{k-1}}\right)\left(\frac{i_{k+1}}{\binom{n}{k+1}}\right)$
- mixing of "down-up" Markov chain

  $g=\sum_{S\in\binom{[n]}{d}} \mu(S)\, x^S$ Walk: $S \xrightarrow{\text{prob } 1/d} S\setminus\{i\} \xrightarrow{\text{prob} \propto \mu(S')} S' = S\setminus\{i\}\cup\{j\}$

Thm (ALOV) Station dist $\propto (\mu(S))_{S \in \binom{[n]}{d}}$

If g is SLC then λ_2(trans. matrix) $\leq 1 - \frac{1}{d}$

$\Rightarrow$ mixing time $O(d^2 \log(n)) \to O(d \log(d))$

builds off [Kauffman-Oppenheim, '18]

- approximate counting via convex optimization (TODAY) via "capacity", "max entropy dist"

Ref: Capacity $\Big\{$
- Gurvits '06: Van der Waerden Conj
- AOV '18: matroids & matroid intersections
- Brändén-Leake-Pak '20: Contigency tables
- Gurvits-Klein-Leake '24: TSP $(3/2 - 10^{-36} \to 3/2 - 10^{-34})$

max entropy: Singh-Vishnoi '15: Entropy, optimization, & counting

For $g \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ define

$$\mathrm{Cap}(g) = \inf_{x \in \mathbb{R}^n_{>0}} \frac{g(x)}{\prod x_i}$$

Thm (Gurvits) If g is SLC, then

$$\frac{1}{e^n} \mathrm{Cap}(g) \leq \partial_1 \cdots \partial_n g \leq \mathrm{Cap}(g)$$

$\frac{n!}{n^n}$ when g homog. of deg n ($\Rightarrow g^{1/n}$ concave)

Ex: $g = x_1^2 + 3x_1x_2 + x_2^2$ $\quad \mathrm{Cap}(g) = \inf_{x>0} \frac{x_1}{x_2} + 3 + \frac{x_2}{x_1} = 5$

(n=2) $\frac{5}{2} = \frac{2!}{2^2} 5 \le \partial_1\partial_2 g = 3 \le 5$

Ex: $A \in \mathbb{R}^{n\times n}_{\ge 0}$, $g = \prod_{i=1}^{n} (Ax)_i = \prod_i \left(\sum_j a_{ij}x_j\right)$

$$\partial_1\cdots\partial_n g = \sum_{\pi\in S_n}\prod_i a_{i\pi(i)} = \mathrm{per}(A)$$

Van der Waerden Conj:

$A\in\mathbb{R}^{n\times n}_{\ge 0}$ doubly stoch. $\Rightarrow \mathrm{per}(A) \ge \frac{n!}{n^n}$

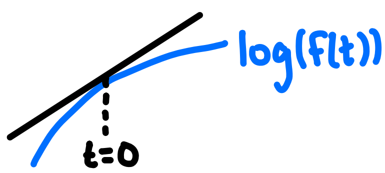
(Proof from Thm): WTS $\mathrm{Cap}(g) = 1$

AM-GM: $g(x) \ge \prod_i\left(\prod_j x_j^{a_{ij}}\right) = \prod_j x_j^{\sum_i a_{ij}} = \prod x_j$

$\Rightarrow \frac{g(x)}{\prod x_j} \ge 1 \quad = 1$ for $x = \mathbb{1}$ $\Rightarrow \inf_{x>0} \frac{g(x)}{\prod x_j} = 1$

Claim: $f\in\mathbb{R}_{\ge 0}[t]$ log-concave $\Rightarrow \inf_{t>0}\frac{f(t)}{t} \le e\cdot f'(0)$

(Proof) Assume $f(0)\neq 0$ and rescale $\to f(0)=1$ (Hwk: what if $f(0)=0$)



$$\log(f(t)) \le \log(f(0)) + t\frac{f'(0)}{f(0)} = tf'(0)$$

$\Rightarrow \inf_{t>0}\frac{f(t)}{t} \le \inf_{t>0}\frac{e^{tf'(0)}}{t} = f'(0)\cdot e \quad \left(t = \frac{1}{f'(0)}\right)$

(replace $\log(f) \to f^{1/k}$ for stronger const.)

(Proof of Thm)(Induct on n) $h = (\partial_{x_n} g)|_{x_n=0}$

$$\inf_{x \in \mathbb{R}^n_{>0}} \frac{g(x)}{x_1 \cdots x_n} = \inf_{x_1,\ldots,x_{n-1}>0} \frac{1}{x_1 \cdots x_{n-1}} \underbrace{\left[\inf_{x_n>0} \frac{g(x)}{x_n} \right]}_{\le e \cdot h}$$

$$\le e \cdot \mathrm{Cap}(h)$$

$$\le e \cdot e^{n-1} \partial_2 \cdots \partial_n h = e^n \partial_1 \cdots \partial_n g \quad \square$$

Linial-Samorodnitsky-Wigderson $O(e^n)$

poly time $O(e^n)$ approx for $\mathrm{per}(A)$

(matrix scaling + VdW bound) $A \in \mathbb{R}^{n \times n}_{\ge 0}$

improved to $\sqrt{2}^n$ (Anari, Rezaei '18)

Matroid intersection

$M = ([n], B_M)$ $N = ([n], B_N)$ matroids

Want to approx. $|B_M \cap B_N|$

(for $N = \binom{[n]}{r}$), this is $|B_M|$)

Special case: #perfect matchings in a bipartite graph

Azar, Broder, Frieze show approx factor

for $|B_M| \ge 2^{O(n/\log(n)^2)}$ $(\to 2^{\Omega(r/\log(n)^2)}$ for $r \gg \log(n))$

for deterministic poly time alg. (indep. oracle)

$$g_M(x) = \sum_{B \in \mathcal{B}_M} x^B \qquad g_{N^*}(y) = \sum_{B \in \mathcal{B}_N} y^{[n] \setminus B}$$

<u>Claim</u>: $|\mathcal{B}_M \cap \mathcal{B}_M| = \prod_{i=1}^{n} (\partial_{x_i} + \partial_{y_i}) \overbrace{g_M(x)\, g_{N^*}(y)}^{G(x,y)}$

(Proof) $\prod_{i=1}^{n} (\partial_{x_i} + \partial_{y_i}) = \sum_{S \subseteq [n]} \partial_x^S \partial_y^{[n] \setminus S}$

$|A| + |B| = n$, $\partial_x^S \partial_y^{[n]\setminus S} x^A y^B = \begin{cases} 1 & \text{if } A = S,\ B = [n]\setminus S \\ 0 & \text{o.w.} \end{cases}$

$\partial_x^S \partial_y^{[n]\setminus S} g_M(x)\, g_{N^*}(y) = \begin{cases} 1 & \text{if } S \in \mathcal{B}_M \cap \mathcal{B}_N \\ 0 & \text{o.w.} \end{cases}$

Lemma: For any $p \in [0,1]^n$,

$$\prod_{i=1}^{n} (\partial_{x_i} + \partial_{y_i}) \cdot G \geq \frac{p^p}{e^2} \underbrace{\inf_{\substack{x>0 \\ y>0}} \frac{G(x,y)}{x^p y^{1-p}}}$$

only interesting for $P_M \cap P_N$

Use "max entropy distribution" to bound

Define $\mathcal{H}(p) := \sum_{i=1}^{n} \left(p_i \log\left(\frac{1}{p_i}\right) + (1-p_i) \log\left(\frac{1}{1-p_i}\right) \right)$

Thm: $\mathcal{H}(p) - 3r \leq \underline{\log |\mathcal{B}_M \cap \mathcal{B}_N|} \leq \mathcal{H}(p)$

any $p \in [0,1]^n$

$p_i = \text{Prob}(i \in B)$ unif. $B \in \mathcal{B}_M \cap \mathcal{B}_N$

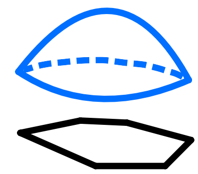
= entropy of unif. dist. on $\mathcal{B}_M \cap \mathcal{B}_N$

(For dist $\mu : 2^{[n]} \to \mathbb{R}_{\geq 0}$, entropy $= \sum_{S \subseteq [n]} \mu(S) \log\left(\frac{1}{\mu(S)}\right)$)

$P_M = \mathrm{conv}\{1_B : B \in \mathcal{B}_M\}$ $\quad P_N := \mathrm{conv}\{1_B : B \in \mathcal{B}_N\}$

Useful fact: $\mathrm{conv}\{1_B : B \in \mathcal{B}_M \cap \mathcal{B}_N\} = P_M \cap P_N$

Compute $\tau = \max\{\mathcal{H}(p) : p \in P_M \cap P_N\}$



$\mathcal{H}(p)$ concave, use ellipsoid method

$\Rightarrow e^{\tau - 3r} \le |\mathcal{B}_M \cap \mathcal{B}_N| \le e^{\tau}$

Thm (AOV '18) This gives det. poly time alg. computing $2^{O(r)}$ mult. approx of $|\mathcal{B}_M \cap \mathcal{B}_N|$