13^{th} Cargèse Workshop in Combinatorial Optimization
Open Problems

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Problem 1 (Kristóf Bérczi)**.** Sampling Common Bases.

Anari et al. [ALGV19] verified that the basis exchange graph of any matroid has edge expansion at least one, and thus gave an efficient approximate sampling algorithm for all matroids using a natural "down-up" random walk which mixes to the uniform distribution over all bases. Sampling common bases of two matroids is also of interest. However, the intersection of two matroids does not satisfy the exchange property, showing that there is no hope for defining a simple down-up-type random walk in general.

Let $\mathcal{X} = (X_1, \dots, X_k)$ be a sequence of – not necessarily disjoint – bases of a matroid, and let $e \in X_i - X_j$ and $f \in X_j - X_i$ with $1 \le i < j \le k$ be such that both $X_i - e + f$ and $X_j + e - f$ are bases. Then, the sequence $\mathcal{X}' = (X_1, \dots, X_{i-1}, X_i - e + f, X_{i+1}, \dots, X_{j-1}, X_j + e - f, X_{j+1}, \dots, X_k)$ is obtained from $\mathcal{X}$ by a symmetric exchange. Two sequences $\mathcal{X}$ and $\mathcal{Y}$ are called equivalent if $\mathcal{Y}$ can be obtained from $\mathcal{X}$ by a composition of symmetric exchanges. Furthermore, we call $\mathcal{X}$ and $\mathcal{Y}$ compatible if $|\{i \mid e \in X_i, i \in \{1, \dots, k\}\}| = |\{i \mid e \in Y_i, i \in \{1, \dots, k\}\}|$ for every $e \in E$, where E denotes the ground set of the matroid. White conjectured the following:

Conjecture (White [Whi80])**.** Two basis sequences $\mathcal{X}$ and $\mathcal{Y}$ of the same length are equivalent if and only if they are compatible.

Partitions of the ground set of a matroid M into two disjoint bases can be identified with common bases of M and its dual M^*. From this perspective, White's conjecture states that there is a sequence of exchanges between any pair of common basis of M and M^*. Thus verifying Conjecture 1 would open up the possibility for a natural down-up random walk for matroid intersection in the special case when the two matroids are dual to each other. This leads to the following problem:

Question. Give an efficient algorithm that samples a uniformly random common basis of M and M^*.

Problem 2 (Jean Cardinal)**.** The edge expansion of a graph $G = (V, E)$ is defined as follows:

$$h(G) = \min_{S \subseteq V} \frac{|E(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}}.$$

Conjecture (Mihail-Vazrani [Kai04])**.** Graphs of $\{0, 1\}$-polytopes have $h \ge 1$.

Theorem ([CP24])**.** Graphs of d-dimensional $\{0, 1/2, 1\}$-polytopes can have $h \le d/\sqrt{2}^d$.

Theorem ([CP24])**.** Graphs of $\{0, 1/2, 1\}$-zonotopes have $h \ge 7/12$.

Question. Do the graphs of $\frac{1}{k}$-integral graphic zonotopes have $h \ge \Omega(f(k))$ for some f?

Problem 3 (Katharina Eickhoff)**.** This problem is motivated by our work on Walrasian prices [ENP⁺23]. Let $f : 2^E \longrightarrow \mathbb{R}$ be a submodular function. Then $\mathcal{L} = \{S \subseteq E \mid f(S) \text{ is minimal}\}$ is a lattice. We are interested in finding how the lattice changes, when f is perturbed slightly. More concrete, if f' is the perturbation of f and $\mathcal{L}'$ the corresponding lattice, we ask the following questions:

1. How does the lattice change? Also, by how much the minimal/maximal element of the lattice changes?

2. Can we find some element $S \in \mathcal{L}$ with minimal distance to $\mathcal{L}'$, i.e.,

$$S = \operatorname*{argmin}_{T \subseteq E} \left(\min_{T' \in \mathcal{L}'} \mathrm{dist}(T, T') \right) ?$$

3. Can we bound the distance of the two lattices?

4. What about the dual solution? In the setting of Walrasian equilibria, the minimizers of the submodular Lyapunov function [Aus06] are the Walrasian prices, and the dual solution is a corresponding allocation.

Problem 4 (Bertrand Guenin)**.** Consider a planar r-regular multigraph G where r is a sufficiently large number. Let $c_1 : E \longrightarrow [r]$ and $c_2 : E \longrightarrow [r]$ be proper edge-colorings. Can we go from c_1 to c_2 by swapping along alternating cycles consisting of two colors? This is not necessarily true for $r = 3$.

Problem 5 (Sharat Ibrahimpur)**.** Matroid Completion Problem.

Given a groundset E and a family of subsets $\mathcal{S}$ of E such that $\bigcup_{S \in \mathcal{S}} S = E$, we are interested in a matroid $M = (E, \mathcal{I})$ such that $\mathcal{S} \subseteq \mathcal{I}$ and $|\mathcal{I}|$ is minimized. More specifically, we want a 'nice' characterization of some minimum-size 'matroid completion'. Is there a polynomial (in $|\mathcal{I}|$) time algorithm to compute this matroid with a full description of its independent sets?

The question can be asked with extra conditions on the matroid, that it be graphic, base orderable, strongly base orderable, linear, etc. Another variant is to restrict $\mathcal{I}$ to disallow any sets larger than $\max_{S \in \mathcal{S}} |S|$ and counting the number of bases.

Problem 6 (Christian Nöbel)**.** Is there a deterministic algorithm for congruency constrained bipartite perfect matching?

To be precise, the problem we consider is the following: We are given a bipartite graph $G = (V, E)$ with a partition of the edges $E = R \dot\cup B$ into red edges R and blue edges B, together with a modulus $m \in \mathbb{Z}_{\geq 1}$. The problem asks to decide if there is a perfect matching M in G with $|M \cap R| \equiv 0 \bmod m$.

A randomized algorithm is known for all moduli (using polynomial identity testing), while a deterministic algorithm is (to the best of my knowledge) only known for $m \in \{1, 2\}$.

Problem 7 (Philipp Pabst)**.** Fare Zone Assignment.

We are given a tree $G = (V, E)$ and a set of commodities (P_i, w_i, u_i), where each $P_i \subseteq E$ is a path in G, w_i denotes the weight of P_i and u_i denotes the "maximum number of allowed cuts on P_i". We want to solve the following problem:

$$\begin{aligned} \max \; & \sum_{e \in E} y_e \cdot w_e \\ \text{s.t.} \; & \sum_{e \in P_i} y_e \leq u_i \quad \forall i \\ & y_e \in \{0, 1\} \quad \forall e \end{aligned}$$

where $w_e = \sum_{\{i \mid e \in P_i\}} w_i$.

In an ongoing work, Lennart Kauther, Sven Müller, Britta Peis, Khai Van Tran and I have shown that this problem is polytime solvable when G is a path and strongly NP-hard when G is a star. There is a 2-approximation on trees when all u_i are even. What else can we say about approximation results on trees? (E.g. for $u_i = 1$, for bicriterial approximation results or for certain subclasses of trees.)

References

[ALGV19] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials ii: high-dimensional walks and an fpras for counting bases of a matroid. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2019, pages 1–12, New York, NY, USA, 2019. Association for Computing Machinery.

[Aus06] Lawrence M Ausubel. An efficient dynamic auction for heterogeneous commodities. *American Economic Review*, 96(3):602–629, May 2006.

[CP24] Jean Cardinal and Lionel Pournin. The expansion of half-integral polytopes. *arXiv preprint arXiv:2402.14343*, 2024.

[ENP+23] Katharina Eickhoff, Meike Neuwohner, Britta Peis, Niklas Rieken, Laura Vargas Koch, and László A. Végh. Faster dynamic auctions via polymatroid sum. 10 2023.

[Kai04] Volker Kaibel. On the expansion of graphs of 0/1-polytopes. In *The Sharpest Cut: The Impact of Manfred Padberg and His Work*, pages 199–216. SIAM, 2004.

[Whi80] Neil L White. A unique exchange property for bases. *Linear Algebra and its applications*, 31:81–91, 1980.